



STABILITY OF THE RECONSTRUCTION DISCONTINUOUS STURM-LIOUVILLE PROBLEM

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ABSTRACT. In this work, we study stability of the inverse spectral problem for the Sturm-Liouville operator $-D^2 + q$ with discontinuity boundary conditions inside a finite closed interval. We use the method which is given by Ryabushko for regular Sturm-Liouville operator in [22] to obtain stability results. These results give a bound for the difference between the spectral functions of associated problems. In addition, we give asymptotic representation of the eigenvalues and a formula for the representation of the norming constants by two spectra.

1. INTRODUCTION

The inverse problems of spectral analysis imply the restoration of a linear operator from some of its spectral characteristics. Such characteristics are spectra, spectral functions, scattering data etc. Inverse problems of Sturm-Liouville operators have been worked in detail by a lot of mathematicians (see [1]-[10]).

Boundary value problems with discontinuity conditions inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural properties. Such problems are connected with discontinuous material properties. Inverse problems for Sturm-Liouville operators with discontinuities inside the interval were investigated in ([11]-[18]).

The local or global predictions mean that ensure a small variation of the potentials under a small variation of the spectral data and vice versa. The study of the local stability is a long-running problem. The study of the uniform stability has been studied relatively recently in [20]. Uniform stability of the problem in question for potentials in the scale of Sobolev spaces W_2^α with $\alpha = -1$, is solved by a different method in [21]. The stability conclusions are rather popular, in fact with entire spectral data. Ryabushko [22] indicated the variation of the difference

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between two spectral functions ρ_1 and ρ_2 when a finite number of eigenvalues coincide. Another result is given by McLaughlin [19]: if the medium values of the potentials are zero, then we have the relation of local diffeomorphism between potentials in $L_2[0, 1]$ and sequences $\{\lambda_n - n^2\pi^2, \rho_n\}$ in $l_2 \times l_2$, where $\{\rho_n\}$ are the norming constants. Marchenko and Maslov dealt with the stability problem for Sturm-Liouville operators in the case of the spectral functions $p_j(\lambda)$ coincide on given interval [23].

The meaning of the stability problem of differential operators is to estimate the difference between the spectral functions, solutions and potentials when a finite number of the eigenvalues of operators overlap. As far as we know, the stability problems for discontinuous Sturm-Liouville operators have not been studied yet. But these type problems for different types of regular and singular operators have been studied by [22]-[25]. Our approach is much more difficult than the method given in [22], because we apply this method for discontinuous Sturm-Liouville problem. The main aim of this study is to show the stability of the reconstruction discontinuous Sturm-Liouville problem from two spectra on $(0, \pi)$.

2. PRELIMINARIES

Consider the following Sturm-Liouville operator L_1 defined by

$$L_1 y = -y'' + q_1(x)y = \lambda y, \quad (2.1)$$

on the interval $0 < x < \pi$ with the boundary conditions

$$\begin{aligned} y'(0, \lambda) - h_1 y(0, \lambda) &= 0, \\ y'(\pi, \lambda) + H y(\pi, \lambda) &= 0, \end{aligned} \quad (2.2)$$

with the jump conditions

$$\begin{aligned} y\left(\frac{\pi}{2} + 0\right) &= \alpha y\left(\frac{\pi}{2} - 0\right) \\ y'\left(\frac{\pi}{2} + 0\right) &= \alpha^{-1} y'\left(\frac{\pi}{2} - 0\right). \end{aligned} \quad (2.3)$$

Consider the second problem defined by equation (2.1) with boundary conditions

$$\begin{aligned} y'(0, \lambda) - h_2 y(0, \lambda) &= 0, \\ y'(\pi, \lambda) + H y(\pi, \lambda) &= 0, \end{aligned} \quad (2.4)$$

with the jump conditions (2.3), where λ is a spectral parameter, $\alpha \neq 1$, $\alpha > 0$, h_1 , h_2 and H are real constants with $h_1 \neq h_2$ and $q_1(x)$ is a real valued function and has bounded derivative in $L_1(0, \pi)$.

Let $\lambda_{1,0} < \lambda_{1,1} < \dots$ and $\mu_{1,0} < \mu_{1,1} < \dots$ be the eigenvalues of the problems (2.1), (2.2), (2.3) and (2.1), (2.3), (2.4), respectively. It is easily seen that numbers λ_n are real and simple. Moreover in the next section they will be proven that the sequences $\{\lambda_{1,n}\}_{n=0}^{\infty}$ and $\{\mu_{1,n}\}_{n=0}^{\infty}$ satisfy the following asymptotic relations for

$n \rightarrow \infty$, by using classical method

$$\sqrt{\lambda_{1,n}} = n + \frac{\alpha^2 h_1 + H + \sigma_1(\pi)/2}{\pi n} + O\left(\frac{1}{n^2}\right), \tag{2.5}$$

$$\sqrt{\mu_{1,n}} = n + \frac{\alpha^2 h_2 + H + \sigma_1(\pi)/2}{\pi n} + O\left(\frac{1}{n^2}\right), \tag{2.6}$$

respectively, where $\sigma_1(x) = \int_{\pi/2}^x q_1(t) dt$.

Let us consider new problems

$$L_2 y = -y'' + q_2(x)y = \lambda y, \quad (0 < x < \pi) \tag{2.7}$$

with the conditions (2.2), (2.3) and (2.3), (2.4), where the real potential $q_2(x)$ has bounded derivative in $L_1(0, \pi)$.

Let the sequences $\{\lambda_{2,n}\}_{n=0}^\infty$ and $\{\mu_{2,n}\}_{n=0}^\infty$ be the sets of eigenvalues of L_2 satisfying conditions (2.2), (2.3) and (2.3), (2.4), respectively. The sequences $\{\lambda_{2,n}\}_{n=0}^\infty$ and $\{\mu_{2,n}\}_{n=0}^\infty$ satisfy the following asymptotic relations for $n \rightarrow \infty$,

$$\sqrt{\lambda_{2,n}} = n + \frac{\alpha^2 h_1 + H + \sigma_2(\pi)/2}{\pi n} + O\left(\frac{1}{n^2}\right), \tag{2.8}$$

and

$$\sqrt{\mu_{2,n}} = n + \frac{\alpha^2 h_2 + H + \sigma_2(\pi)/2}{\pi n} + O\left(\frac{1}{n^2}\right), \tag{2.9}$$

where $\sigma_2(x) = \int_{\pi/2}^x q_2(t) dt$.

Let the functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the solutions of (2.1) and (2.7), respectively, which satisfy the initial conditions

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h_1 \tag{2.10}$$

and the jump conditions (2.3). It can be proven that $\varphi(\lambda, x)$ is also the solution of the following integral equations:

For $x < \frac{\pi}{2}$

$$\varphi(x, \lambda) = \cos \sqrt{\lambda} x + \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda} x + \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda}(x-t) q_1(t) \varphi(t, \lambda) dt; \tag{2.11}$$

for $x > \frac{\pi}{2}$

$$\begin{aligned} \varphi(x, \lambda) = & \alpha^+ \left(\cos \sqrt{\lambda}x + \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}x \right) \\ & + \alpha^- \left(\cos \sqrt{\lambda}(\pi - x) + \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(\pi - x) \right) \\ & + \frac{\alpha^+}{\sqrt{\lambda}} \int_0^{\frac{\pi}{2}} \sin \sqrt{\lambda}(x - t) q_1(t) \varphi(t, \lambda) dt \\ & + \frac{\alpha^-}{\sqrt{\lambda}} \int_0^{\frac{\pi}{2}} \sin \sqrt{\lambda}(\pi - x - t) q_1(t) \varphi(t, \lambda) dt \\ & + \frac{1}{\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x \sin \sqrt{\lambda}(x - t) q_1(t) \varphi(t, \lambda) dt \end{aligned} \tag{2.12}$$

where

$$\alpha^\pm = \frac{1}{2} \left(\alpha \pm \frac{1}{\alpha} \right).$$

Set the norming constants and spectral functions of the problems (2.1), (2.2), (2.3) and (2.2), (2.3), (2.7) by

$$\alpha_{1,n} = \int_0^{\frac{\pi}{2}-0} \varphi^2(x, \lambda_{1,n}) dx + \int_{\frac{\pi}{2}+0}^{\pi} \varphi^2(x, \lambda_{1,n}) dx, \quad p_1(\lambda) = \sum_{\lambda_{1,n} < \lambda} \frac{1}{\alpha_{1,n}}$$

and

$$\alpha_{2,n} = \int_0^{\frac{\pi}{2}-0} \psi^2(x, \lambda_{2,n}) dx + \int_{\frac{\pi}{2}+0}^{\pi} \psi^2(x, \lambda_{2,n}) dx, \quad p_2(\lambda) = \sum_{\lambda_{2,n} < \lambda} \frac{1}{\alpha_{2,n}},$$

respectively.

3. MAIN RESULTS

In this study, we apply Ryabushko’s method given in [22] for Sturm-Liouville operator with discontinuity conditions inside an interval. By this method, we obtain a bound for variation of the spectral functions $p_1(\lambda)$ and $p_2(\lambda)$ when the eigenvalues $\{\lambda_{j,n}\}$ and $\{\mu_{j,n}\}$ coincide the numbers of $N + 1$ for $n = 1, 2, \dots, N + 1$. In addition, we derive a formula for the norming constants of problem (2.1), (2.2), (2.3) with respect to two spectra.

Theorem 1. *Following equality holds*

$$\alpha_{1,n} = \frac{h_2 - h_1}{\mu_{1,n} - \lambda_{1,n}} \prod_{k=0}^{\infty} \frac{\lambda_{1,k} - \lambda_{1,n}}{\mu_{1,k} - \lambda_{1,n}}$$

for $n \in \mathbb{N}$, where the symbol \prod' means that the factor with the number $k = n$ has been omitted from the infinite product.

Proof. The functions $\varphi(x, \lambda)$ and $\zeta(x, \lambda)$ are the solutions of equation (2.1) satisfying conditions

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h_1 \quad (3.1)$$

$$\zeta(0, \lambda) = 1, \quad \zeta'(0, \lambda) = h_2, \quad (3.2)$$

respectively. The eigenvalues $\{\lambda_{1,n}\}$ and $\{\mu_{1,n}\}$ coincide with the zeros of the functions

$$\Phi_1(\lambda) = \varphi'(\pi, \lambda) + H\varphi(\pi, \lambda),$$

$$\Phi_2(\lambda) = \zeta'(\pi, \lambda) + H\zeta(\pi, \lambda),$$

respectively. In this proof, for briefly denote by $\lambda_{1,n} = \lambda_n$ and $\mu_{1,n} = \mu_n$. The functions $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are entire in λ for fixed x . It's clear from that $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are entire functions of order one half and therefore are determined by their zeros, to within a constant multiplying factor. Therefore,

$$\Phi_1(\lambda) = C_1 \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right)$$

and

$$\Phi_2(\lambda) = C_2 \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\mu_k}\right)$$

where C_1 and C_2 are constants. Put

$$f(x, \lambda) = \zeta(x, \lambda) + m(\lambda)\varphi(x, \lambda)$$

and $f(x, \lambda)$ satisfy the boundary condition

$$f'(\pi, \lambda) + Hf(\pi, \lambda) = 0. \quad (3.3)$$

It follows from (3.3)

$$m(\lambda) = -\frac{\zeta'(\pi, \lambda) + H\zeta(\pi, \lambda)}{\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda)} = -\frac{\Phi_2(\lambda)}{\Phi_1(\lambda)}.$$

It can be seen from this formula that $m(\lambda)$ is a meromorphic function, its poles and zeros coinciding with the eigenvalues of problem (2.1), (2.2), (2.3) and (2.1),

(2.3), (2.4), respectively. It follows from Green's formula that

$$\begin{aligned}
& (\lambda_1 - \lambda_2) \int_0^{\frac{\pi}{2}-0} f(x, \lambda_1) f(x, \lambda_2) dx + (\lambda_1 - \lambda_2) \int_{\frac{\pi}{2}+0}^{\pi} f(x, \lambda_1) f(x, \lambda_2) dx \\
= & \int_0^{\frac{\pi}{2}-0} \{-f''(x, \lambda_1) f(x, \lambda_2) + f(x, \lambda_1) f''(x, \lambda_2)\} dx + \\
& + \int_{\frac{\pi}{2}+0}^{\pi} \{-f''(x, \lambda_1) f(x, \lambda_2) + f(x, \lambda_1) f''(x, \lambda_2)\} dx \\
= & (f'(x, \lambda_2) f(x, \lambda_1) - f'(x, \lambda_1) f(x, \lambda_2)) \Big|_0^{\frac{\pi}{2}-0} \\
& + (f'(x, \lambda_2) f(x, \lambda_1) - f'(x, \lambda_1) f(x, \lambda_2)) \Big|_{\frac{\pi}{2}+0}^{\pi} \\
= & [m(\lambda_1) - m(\lambda_2)] (h_1 - h_2). \tag{3.4}
\end{aligned}$$

If we get $\lambda_1 = \lambda$, $\lambda_2 = \bar{\lambda}$ in (3.4), we obtain

$$\int_0^{\frac{\pi}{2}-0} f(x, \lambda) f(x, \bar{\lambda}) dx + \int_{\frac{\pi}{2}+0}^{\pi} f(x, \lambda) f(x, \bar{\lambda}) dx = (h_1 - h_2) \frac{Imm(\lambda)}{Im\lambda}.$$

It can be seen from this formula that if $h_2 > h_1$, then the function $m(\lambda)$ maps the upper half-plane onto itself (for $h_2 < h_1$ this holds for the lower half-plane). Hence, the zeros and poles of the function $m(\lambda)$, i.e. the eigenvalues of problems (2.1), (2.2), (2.3) and (2.1), (2.3), (2.4) alternate. Applying Green's formula again, we have

$$\begin{aligned}
& (\lambda - \lambda_n) \int_0^{\frac{\pi}{2}-0} f(x, \lambda) \varphi(x, \lambda_n) dx + (\lambda - \lambda_n) \int_{\frac{\pi}{2}+0}^{\pi} f(x, \lambda) \varphi(x, \lambda_n) dx \\
= & (f(x, \lambda) \varphi'(x, \lambda_n) - f'(x, \lambda) \varphi(x, \lambda_n)) \Big|_0^{\frac{\pi}{2}-0} \\
& + (f(x, \lambda) \varphi'(x, \lambda_n) - f'(x, \lambda) \varphi(x, \lambda_n)) \Big|_{\frac{\pi}{2}+0}^{\pi} \\
= & h_2 - h_1.
\end{aligned}$$

Assuming that $\lambda \rightarrow \lambda_n$, we have the formula

$$\alpha_n = \int_0^{\frac{\pi}{2}-0} \varphi^2(x, \lambda_n) dx + \int_{\frac{\pi}{2}+0}^{\pi} \varphi^2(x, \lambda_n) dx = \frac{h_2 - h_1}{\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) m(\lambda_n)}. \tag{3.5}$$

The distribution of the zeros of entire function $m(\lambda_n)$ as follows:

$$\begin{aligned}
m(\lambda_n) &= -\frac{C_2}{C_1} \prod_{k=0}^{\infty} \left(1 - \frac{\lambda_n}{\mu_k}\right) \left(1 - \frac{\lambda_n}{\lambda_k}\right)^{-1} \\
&= -\frac{C_2}{C_1} \prod_{k=0}^{\infty} \left(\frac{\mu_k}{\lambda_k}\right) \prod_{k=0}^{\infty} \left(\frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}\right)
\end{aligned}$$

here we must show the equality $C = \frac{C_2}{C_1} \prod_{k=0}^{\infty} \left(\frac{\mu_k}{\lambda_k} \right) = 1$ [2]. The asymptotics of the solutions yield that

$$\lim_{\lambda \rightarrow -\infty} \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} = 1,$$

that is

$$\begin{aligned} & \lim_{\lambda \rightarrow -\infty} \frac{C_1}{C_2} \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_k} \right) \left(1 - \frac{\lambda}{\mu_k} \right)^{-1} \\ &= \frac{C_1}{C_2} \prod_{k=0}^{\infty} \left(\frac{\mu_k}{\lambda_k} \right) \lim_{\lambda \rightarrow -\infty} \prod_{k=0}^{\infty} \left(\frac{\lambda_k - \lambda}{\mu_k - \lambda} \right) = 1 \end{aligned} \quad (3.6)$$

It can be easily seen that

$$\lim_{\lambda \rightarrow -\infty} \prod_{k=0}^{\infty} \left(\frac{\lambda_k - \lambda}{\mu_k - \lambda} \right) = \lim_{\lambda \rightarrow -\infty} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda_k - \mu_k}{\mu_k - \lambda} \right). \quad (3.7)$$

Now, considering the following series

$$\sum_{k=0}^{\infty} \frac{\lambda_k - \mu_k}{\mu_k - \lambda}$$

and using the asymptotic formulas of the eigenvalues in (2.5) and (2.6), we obtain $\lambda_k - \mu_k = O(1)$ and the series

$$\sum_{k=1}^{\infty} \frac{\lambda_k - \mu_k}{\mu_k - \lambda} + \frac{\lambda_0 - \mu_0}{\mu_0 - \lambda}$$

converges uniformly in a neighbourhood of the point $\lambda = -\infty$. Therefore the limit in each term of the infinite product (3.7) can be approached, that is,

$$\lim_{\lambda \rightarrow -\infty} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda_k - \mu_k}{\mu_k - \lambda} \right) = 1.$$

Considering (3.6) and (3.7), we show that $C = 1$. Then we can rewrite (3.5) as

$$\alpha_n = - \frac{h_2 - h_1}{\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n)} \prod_{k=0}^{\infty} \frac{\lambda_k - \lambda_n}{\mu_k - \lambda_n}$$

where the factor with the number $k = n$ has been omitted from the infinite product which denotes the symbol \prod' .

Hence, we denote the representation of the norming constants of problem (2.1), (2.2), (2.3) with respect to two spectra by

$$\alpha_{1,n} = \frac{h_2 - h_1}{\mu_{1,n} - \lambda_{1,n}} \prod_{k=0}^{\infty} \frac{\lambda_{1,k} - \lambda_{1,n}}{\mu_{1,k} - \lambda_{1,n}}$$

for every $n \in \mathbb{N}$. So, the proof is completed.

Next we give main theorem in this study. □

Theorem 2. *Let the eigenvalues $\{\lambda_{1,k}\}$, $\{\lambda_{2,k}\}$ and $\{\mu_{1,k}\}$, $\{\mu_{2,k}\}$ coincide numbers of $N + 1$ with each other, that is, $\lambda_{1,k} = \lambda_{2,k}$ and $\mu_{1,k} = \mu_{2,k}$ for $k = 1, 2, \dots, N + 1$, then*

$$\text{Var}_{-\infty < \lambda < (\frac{N}{2})^2} \{\rho_1(\lambda) - \rho_2(\lambda)\} < \rho_1 \left(\frac{N^2}{4} \right) \frac{8A \left(1 + \frac{3}{2N^2} \right)}{3N^2} e^{\frac{3A(1 + \frac{3}{2N^2})}{N^2}},$$

for $k > N + 1$, $n < \frac{N}{2}$, and $N \geq 3\sqrt{A}$ where

$$A = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} |q_2(t) - q_1(t)| dt + O\left(\frac{1}{k^2}\right).$$

Before giving the proof of the main theorem let us give some necessary lemmas.

Lemma 3. *Let $q_1'(x) \in L_1(0, \pi)$ and $\text{Im } \lambda \geq 0$, then the following inequalities hold for $\sqrt{\lambda} > \sigma(x)$*

$$\left| \left(\varphi(\lambda, x) - \cos \sqrt{\lambda}x - \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}x \right) e^{i\sqrt{\lambda}x} \right| \leq \frac{\sigma(x) \left(1 + \frac{h}{\sqrt{\lambda}} \right)}{|\sqrt{\lambda}| - \sigma(x)}, \quad x < \frac{\pi}{2} \quad (3.8)$$

$$\begin{aligned} & \left| \left(\varphi(\lambda, x) - (\alpha^+ - \alpha^-) \cos \sqrt{\lambda}x - (\alpha^+ + \alpha^-) \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}x \right) e^{i\sqrt{\lambda}x} \right| < \\ & \frac{2\alpha^2 \sigma\left(\frac{\pi}{2}\right) |\lambda| \left(1 + \frac{h_1}{\sqrt{\lambda}} \right) + 2 \left(|\sqrt{\lambda}| - \sigma(x) \right) \sigma_{11}(x) \left(\alpha^2 h_1 + |\sqrt{\lambda}| \right)}{|\alpha\sqrt{\lambda}| \left(|\sqrt{\lambda}| - \sigma(x) \right) \left(|\sqrt{\lambda}| - \sigma_{11}(x) \right)}, \quad x > \frac{\pi}{2} \quad (3.9) \end{aligned}$$

$$\begin{aligned} & \left| \tilde{\tau}(\lambda, x) - \frac{(\alpha^+ - \alpha^-) e^{i\sqrt{\lambda}x} \sin \sqrt{\lambda}x}{2\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x q_1(t) dt \right| \\ & < \frac{(\alpha^+ + \alpha^-) \sigma\left(\frac{\pi}{2}\right) + \frac{(\alpha^+ - \alpha^-)}{2|\sqrt{\lambda}|} \sigma_{11}(x) + O\left(\frac{1}{\lambda}\right)}{|\sqrt{\lambda}|}, \quad x > \frac{\pi}{2} \quad (3.10) \end{aligned}$$

where $\sigma(x) = \int_0^x |q_1(t)| dt$ and $\sigma_{11}(x) = \int_{\frac{\pi}{2}}^x |q_1(t)| dt$.

Proof. For $x < \frac{\pi}{2}$, consider the function defined by

$$\tau(\lambda, x) = \left[\varphi(\lambda, x) - \cos \sqrt{\lambda}x - \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}x \right] e^{i\sqrt{\lambda}x}, \quad \text{Im } \sqrt{\lambda} \geq 0.$$

It is clear that $\tau(\lambda, x)$ can be written as follows:

$$\begin{aligned} \tau(\lambda, x) &= \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda}(x-t) q_1(t) \left[\tau(\lambda, t) e^{-i\sqrt{\lambda}t} \right] e^{i\sqrt{\lambda}x} dt \\ &+ \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda}(x-t) q_1(t) \cos \sqrt{\lambda}t e^{i\sqrt{\lambda}(x-t)} e^{i\sqrt{\lambda}t} dt \\ &+ \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda}(x-t) q_1(t) \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}t e^{i\sqrt{\lambda}(x-t)} e^{i\sqrt{\lambda}t} dt. \end{aligned} \quad (3.11)$$

Let introduce

$$m(\lambda, x) = \max_{0 \leq t \leq x} |\tau(\lambda, t)|.$$

It is known that by [22], we have

$$\begin{aligned} \left| \frac{\sin \sqrt{\lambda}(x-t) e^{i\sqrt{\lambda}(x-t)}}{\sqrt{\lambda}} \right| &\leq \frac{1}{\sqrt{\lambda}}, \quad (\operatorname{Im} \lambda \geq 0, x \geq t \geq 0), \\ \left| \cos \sqrt{\lambda}t e^{i\sqrt{\lambda}t} \right| &\leq 1, \quad (\operatorname{Im} \lambda \geq 0, t \geq 0). \end{aligned} \quad (3.12)$$

The equation (3.12) yields that

$$m(\lambda, x) \leq \frac{m(\lambda, x)}{|\sqrt{\lambda}|} \int_0^x |q_1(t)| dt + \frac{1}{|\sqrt{\lambda}|} \int_0^x |q_1(t)| dt + \frac{h_1}{|\lambda|} \int_0^x |q_1(t)| dt$$

Hence, the last inequality gives the equation (3.8) for $|\sqrt{\lambda}| > \sigma(x)$. Let us prove equation (3.9). Let's define the function for $x > \frac{\pi}{2}$, by (2.12)

$$\tilde{\tau}(\lambda, x) = \left[\varphi(\lambda, x) - (\alpha^+ - \alpha^-) \cos \sqrt{\lambda}x - \frac{h_1(\alpha^+ + \alpha^-)}{\sqrt{\lambda}} \sin \sqrt{\lambda}x \right] e^{i\sqrt{\lambda}x}, \quad \operatorname{Im} \sqrt{\lambda} \geq 0.$$

It is clear that $\tilde{\tau}(\lambda, x)$ can be written as

$$\begin{aligned} \tilde{\tau}(\lambda, x) &= \frac{\alpha^+}{\sqrt{\lambda}} \int_0^{\frac{\pi}{2}} \sin \sqrt{\lambda}(x-t) q_1(t) \left[\tau(\lambda, t) e^{-i\sqrt{\lambda}t} + \cos \sqrt{\lambda}t + \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}t \right] e^{i\sqrt{\lambda}x} dt \\ &+ \frac{\alpha^-}{\sqrt{\lambda}} \int_0^{\frac{\pi}{2}} \sin \sqrt{\lambda}(x+t) q_1(t) \left[\tau(\lambda, t) e^{-i\sqrt{\lambda}t} + \cos \sqrt{\lambda}t + \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}t \right] e^{i\sqrt{\lambda}x} dt \\ &+ \frac{1}{\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x \sin \sqrt{\lambda}(x-t) q_1(t) \tilde{\tau}(\lambda, t) e^{-i\sqrt{\lambda}t} e^{i\sqrt{\lambda}x} dt \\ &+ \frac{(\alpha^+ - \alpha^-)}{\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x \sin \sqrt{\lambda}(x-t) q_1(t) \cos \sqrt{\lambda}t e^{i\sqrt{\lambda}x} dt \\ &+ \frac{(\alpha^+ + \alpha^-)}{\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x \sin \sqrt{\lambda}(x-t) q_1(t) \frac{h_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}t e^{i\sqrt{\lambda}x} dt. \end{aligned} \tag{3.13}$$

Denote

$$\tilde{m}(\lambda, x) = \max_{\frac{\pi}{2} \leq t \leq x} |\tilde{\tau}(\lambda, t)|.$$

Considering last equality with (3.12), we can rewrite (3.13) as:

$$\begin{aligned} \tilde{m}(\lambda, x) &\leq \frac{(\alpha^+ + \alpha^-) m(\lambda, x)}{|\sqrt{\lambda}|} \int_0^{\frac{\pi}{2}} |q_1(t)| dt + \frac{\alpha^+}{|\sqrt{\lambda}|} \int_0^{\frac{\pi}{2}} |q_1(t)| dt \\ &+ \frac{(\alpha^+ + \alpha^-) h_1}{|\lambda|} \int_0^{\frac{\pi}{2}} |q_1(t)| dt + \frac{(\alpha^+ + \alpha^-)}{|\sqrt{\lambda}|} \int_0^{\frac{\pi}{2}} |q_1(t)| dt + \\ &+ \frac{(\alpha^+ - \alpha^-)}{|\sqrt{\lambda}|} \int_{\frac{\pi}{2}}^x |q_1(t)| dt + \frac{(\alpha^+ + \alpha^-) h_1}{|\lambda|} \int_{\frac{\pi}{2}}^x |q_1(t)| dt \\ &+ \frac{\tilde{m}(\lambda, x)}{|\sqrt{\lambda}|} \int_{\frac{\pi}{2}}^x |q_1(t)| dt. \end{aligned} \tag{3.14}$$

Writing (3.8) into (3.14), desired result is obtained. Now let's consider the integral to obtain (3.10)

$$\begin{aligned}
& \frac{(\alpha^+ - \alpha^-)}{\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x \sin \sqrt{\lambda}(x-t) \cos \sqrt{\lambda}t q_1(t) dt \\
& + \frac{h_1(\alpha^+ + \alpha^-)}{\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x \sin \sqrt{\lambda}(x-t) \sin \sqrt{\lambda}t q_1(t) dt \\
= & \frac{(\alpha^+ - \alpha^-) \sin \sqrt{\lambda}x}{2\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x q_1(t) dt - \frac{(\alpha^+ + \alpha^-) h_1 \cos \sqrt{\lambda}x}{2\lambda} \int_{\frac{\pi}{2}}^x q_1(t) dt \\
& + \frac{(\alpha^+ - \alpha^-)}{2\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x \sin \sqrt{\lambda}(x-2t) q_1(t) dt \\
& + \frac{(\alpha^+ + \alpha^-) h_1}{2\lambda} \int_{\frac{\pi}{2}}^x \cos \sqrt{\lambda}(x-2t) q_1(t) dt. \tag{3.15}
\end{aligned}$$

Combining (3.15) with (3.9), we have

$$\begin{aligned}
& \left| \tilde{\tau}(\lambda, x) - \frac{(\alpha^+ - \alpha^-) e^{i\sqrt{\lambda}x} \sin \sqrt{\lambda}x}{2\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x q_1(t) dt \right| \\
< & \frac{(\alpha^+ + \alpha^-)}{\sqrt{\lambda}} \int_0^{\frac{\pi}{2}} |q_1(t)| \tau(\lambda, t) dt + \frac{(\alpha^+ + \alpha^-)}{\sqrt{\lambda}} \int_0^{\frac{\pi}{2}} |q_1(t)| dt \\
& + \frac{(\alpha^+ + \alpha^-) h_1}{\lambda} \int_0^{\frac{\pi}{2}} |q_1(t)| dt + \frac{1}{\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x |q_1(t)| \tilde{\tau}(\lambda, t) dt \\
& + \frac{(\alpha^+ + \alpha^-)}{2\lambda} \int_{\frac{\pi}{2}}^x |q_1(t)| dt + \frac{(\alpha^+ + \alpha^-) h_1}{2\lambda} \int_{\frac{\pi}{2}}^x |q_1(t)| dt \\
& + \frac{(\alpha^+ - \alpha^-) h_1}{2\sqrt{\lambda}} \int_{\frac{\pi}{2}}^x |q_1(t)| dt. \tag{3.16}
\end{aligned}$$

Substituting (3.8) and (3.9) into (3.16), we obtain the inequality (3.10). So the proof is completed. \square

Lemma 4. *The eigenvalues $\lambda_{1,n}$ of problem (2.1), (2.2), (2.3) satisfy the asymptotic formula*

$$\sqrt{\lambda_{1,n}} = n + \frac{\alpha^2 h_1 + H}{\pi n} + \frac{\sigma_1(\pi)}{2\pi n} + O\left(\frac{1}{n^2}\right), \tag{3.17}$$

as $n \rightarrow \infty$.

Proof. From (3.10), one can easily see that

$$\begin{aligned} \varphi(\lambda, x) &= (\alpha^+ - \alpha^-) \cos \sqrt{\lambda}x + \frac{h_1(\alpha^+ + \alpha^-)}{\sqrt{\lambda}} \sin \sqrt{\lambda}x \\ &+ \frac{(\alpha^+ - \alpha^-) \sin \sqrt{\lambda}x}{2\sqrt{\lambda}} \sigma_1(x) + \frac{(\alpha^+ + \alpha^-)}{\sqrt{\lambda}} \sigma\left(\frac{\pi}{2}\right) \\ &+ \frac{(\alpha^+ - \alpha^-)}{2\sqrt{\lambda}} \sigma_{11}(x) + O\left(\frac{1}{\lambda}\right) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \varphi'(\lambda, x) &= h_1(\alpha^+ + \alpha^-) \cos \sqrt{\lambda}x - \sqrt{\lambda}(\alpha^+ - \alpha^-) \sin \sqrt{\lambda}x + \\ &\frac{(\alpha^+ - \alpha^-) \sin \sqrt{\lambda}x q_1(x)}{2\sqrt{\lambda}} + \frac{(\alpha^+ - \alpha^-) q_1(x)}{2\sqrt{\lambda}} \\ &+ \frac{(\alpha^+ - \alpha^-) \cos \sqrt{\lambda}x \sigma_1(x)}{2} + O\left(\frac{1}{\lambda}\right). \end{aligned} \tag{3.19}$$

Putting (3.18) and (3.19) into boundary condition

$$\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda) = 0$$

and making classical calculations, we can easily obtain (3.17).

Now we give the proof of the main theorem in this study. □

Proof of the Main Theorem. Consider the difference between the spectral functions

$$\rho_1(\lambda) - \rho_2(\lambda) = \sum_{\lambda_n < \lambda} \frac{1}{\alpha_{1,n}} \left(1 - \frac{\alpha_{1,n}}{\alpha_{2,n}}\right)$$

where

$$1 - \frac{\alpha_{1,n}}{\alpha_{2,n}} = 1 - \prod_{k=N+2}^{\infty} \frac{(\lambda_{1,k} - \lambda_{1,n})(\mu_{2,k} - \lambda_{2,n})}{(\lambda_{2,k} - \lambda_{2,n})(\mu_{1,k} - \lambda_{1,n})}.$$

From the definition of the variation, we have

$$\begin{aligned} \underset{-\infty < \lambda < \lambda_0}{Var} \{\rho_1(\lambda) - \rho_2(\lambda)\} &\leq \max_{\lambda_n < \lambda_0} \left|1 - \frac{\alpha_{1,n}}{\alpha_{2,n}}\right| \sum_{\lambda_n < \lambda_0} \frac{1}{\alpha_{1,n}} \\ &= \rho_1(\lambda_0) \max_{\lambda_n < \lambda_0} \left|1 - \frac{\alpha_{1,n}}{\alpha_{2,n}}\right| \end{aligned} \tag{3.20}$$

for $\lambda_0 < \lambda_{N+2}$. Therefore, let's consider the absolute value of the term at the right side of (3.20) to assess the difference between the spectral functions as follow:

$$\max_{n < \frac{N}{2}} \left| 1 - \frac{\alpha_{1,n}}{\alpha_{2,n}} \right| = \max_{n < \frac{N}{2}} \left| 1 - \prod_{k=N+2}^{\infty} \frac{(\lambda_{1,k} - \lambda_{1,n})(\mu_{2,k} - \lambda_{2,n})}{(\lambda_{2,k} - \lambda_{2,n})(\mu_{1,k} - \lambda_{1,n})} \right|. \quad (3.21)$$

Considering the infinite product

$$\Psi(\lambda_n) = \prod_{k=N+2}^{\infty} \frac{(\lambda_{1,k} - \lambda_{1,n})(\mu_{2,k} - \lambda_{2,n})}{(\lambda_{2,k} - \lambda_{2,n})(\mu_{1,k} - \lambda_{1,n})},$$

it follows that

$$\begin{aligned} |\ln \Psi(\lambda_n)| &= \left| \sum_{k=N+2}^{\infty} \ln \left(\frac{\lambda_{1,k} - \lambda_{1,n}}{\lambda_{2,k} - \lambda_{2,n}} \right) + \sum_{k=N+2}^{\infty} \ln \left(\frac{\mu_{2,k} - \lambda_{2,n}}{\mu_{1,k} - \lambda_{1,n}} \right) \right| \\ &\leq \sum_{k=N+2}^{\infty} \left| \ln \left(1 - \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}} \right) \right| \\ &\quad + \sum_{k=N+2}^{\infty} \left| \ln \left(1 - \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{1,n}} \right) \right|. \end{aligned}$$

It can be easily seen

$$\left| \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}} \right| < 1 \quad \text{and} \quad \left| \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{1,n}} \right| < 1$$

for $k > N + 1$, $n < \frac{N}{2}$. It is well known that the following inequality holds

$$\ln(1 - z) \leq \frac{|z|}{1 - |z|}$$

for $|z| < 1$. The last inequality implies that

$$|\ln \Psi(\lambda_n)| < \sum_{k=N+2}^{\infty} \frac{\left| \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}} \right|}{1 - \left| \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}} \right|} + \sum_{k=N+2}^{\infty} \frac{\left| \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{1,n}} \right|}{1 - \left| \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{1,n}} \right|}. \quad (3.22)$$

Here, by the asymptotic estimates of the eigenvalues, we can obtain

$$\begin{aligned} \left| \frac{\lambda_{2,k} - \lambda_{1,k}}{\lambda_{2,k} - \lambda_{1,n}} \right| &= \left| \frac{A}{\lambda_{2,k} \left(1 - \frac{\lambda_{1,n}}{\lambda_{2,k}} \right)} \right| \\ &< \frac{A}{\left(k - \frac{1}{2} \right)^2 \left(1 - \frac{\lambda_{1, \lfloor \frac{N}{2} \rfloor}}{\lambda_{2, N+2}} \right)} \\ &= \frac{A}{\left(k - \frac{1}{2} \right)^2 \left(1 - \frac{\left(\frac{N+1}{2} \right)^2}{\left(N+2 - \frac{1}{2} \right)^2} \right)} \\ &< \frac{4A \left(1 + \frac{3}{2N^2} \right)}{3N^2} \end{aligned} \tag{3.23}$$

and

$$\left| \frac{\mu_{1,k} - \mu_{2,k}}{\mu_{1,k} - \lambda_{1,n}} \right| < \frac{4A \left(1 + \frac{3}{2N^2} \right)}{3N^2} \tag{3.24}$$

for $k > N + 1$, $n < \frac{N}{2}$. Substituting (3.23) and (3.24) into (3.22), we have

$$|\ln \Psi(\lambda_n)| < 2 \frac{\frac{4A \left(1 + \frac{3}{2N^2} \right)}{3N^2}}{1 - \frac{4A \left(1 + \frac{3}{2N^2} \right)}{3N^2}}.$$

Considering the last inequality for $N \geq 3\sqrt{A}$, it can be obtained that

$$|\ln \Psi(\lambda_n)| < \frac{8A \left(1 + \frac{3}{2N^2} \right)}{3N^2} \tag{3.25}$$

If we put (3.25) into (3.21) and use the serial expansion of the exponential function, the following inequality holds

$$\begin{aligned} \max_{n < \frac{N}{2}} \left| 1 - \frac{\alpha_{1,n}}{\alpha_{2,n}} \right| &< e^{\frac{8A \left(1 + \frac{3}{2N^2} \right)}{3N^2}} - 1 \\ &< \frac{8A \left(1 + \frac{3}{2N^2} \right)}{3N^2} e^{\frac{3A \left(1 + \frac{3}{2N^2} \right)}{N^2}} \end{aligned} \tag{3.26}$$

for $N \geq 3\sqrt{A}$. Finally, if we put the inequality (3.26) into (3.20), it yields that

$$\text{Var}_{-\infty < \lambda < \lambda_0} \{ \rho_1(\lambda) - \rho_2(\lambda) \} < \rho_1 \left(\frac{N^2}{4} \right) \frac{8A \left(1 + \frac{3}{2N^2} \right)}{3N^2} e^{\frac{3A \left(1 + \frac{3}{2N^2} \right)}{N^2}}$$

for $N \geq 3\sqrt{A}$. Therefore the proof is completed. □

Conclusion 5. *In this study, we have emphasized the importance of the certain stability of the inverse discontinuous Sturm-Liouville problem. Applying Ryabushko's method, we show the proximity of the difference between the spectral functions*

of the problems (2.1), (2.2), (2.3) and (2.2), (2.3), (2.7) whose eigenvalues $\{\lambda_{j,k}\}$ and $\{\mu_{j,k}\}$, coincide the numbers of $N + 1$. We give some asymptotic estimates for the eigenvalues and the eigenfunctions of discontinuous eigenvalue problem (2.1), (2.2), (2.3) and give a formula determining the norming constants according to two spectra.

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