GU J Sci 30(1): 333-343 (2017)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Integral Operators of the Normalized Wright Functions and their Some Geometric Properties

Nizami MUSTAFA[1,*]

[1] *Department of Mathematics, Kafkas University, Kars, Turkey*

Article Info

Received: 16/06/2016
Accepted: 14/12/2016

Keywords

Wright function
Starlike function
Convex function

Abstract

The purpose of the present paper is to investigate some characterization for the functions represented with normalized Wright functions to be in the subclasses $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$, $\alpha,\beta\in[0,1)$ of analytic functions in the open unit disk. Sufficient conditions are obtained for these functions to be in these classes.

1. INTRODUCTION

It is well known that the special functions play an important role in geometric function theory. Furthermore, this is also well known that the application area of the special functions is not limited to the theory of geometric functions. These functions are also wide range of applications in many problems as well as other branches of mathematics and applied sciences. In this paper, we will examine Wright function, which defined by the series

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\lambda n+\mu)}\frac{z^n}{n!},\ \lambda>-1,\mu,z\in\mathbb{C}\,. \tag{1.1}$$

This series is absolutely convergent in $\mathbb{C}$, when $\lambda>-1$ and absolutely convergent in open unit disk for $\lambda=-1$. Furthermore, for $\lambda>-1$, the Wright function $W_{\lambda,\mu}(z)$ is an entire function. The Wright functions were introduced by Wright in [1], and have appeared for the first time in the case $\lambda>0$ in connection with his investigations in the asymptotic theory of partitions. Later on, it has found many other applications, first of all, in the Mikusinski operational calculus and in the theory of integral transforms of Hankel type. Furthermore, extending the methods of Lie groups in partial differential equations to the partial differential equations of fractional order it was shown that some of the group-invariant solutions of these equations can be given in terms of the Wright functions. Recently, Wright functions have appeared in papers related to partial differential equations of fractional order, it was found that the corresponding Green functions can be represented in terms of the Wright functions [2, 3]. A series of paper are devoted to the application of the Wright functions in partial differential equation of fractional order extending the classical diffusion and wave equations. Mainardi [4] has obtained the result for a fractional diffusion wave equation in terms of the fractional Green function involving the Wright function. The scale-variant solutions of some partial differential equations of fractional order were obtained in terms of special cases of the generalized Wright function by Buckwar and Luchko [5] and Luchko and Gorenflo [6].

Several researchers studied classes of analytic functions involving special functions $F\subset A$, to find different conditions such that the members of F have certain geometric properties such as starlkeness and convexity in U. There is an extensive literature dealing with geometric properties of different types of

*Corresponding author, e-mail: nizamimustafa@gmail.com

hypergeometric functions, especially for generalized Gaussian, Kummer hypergeometric and Bessel functions [7-12].

As known, if λ is a positive rational number, then the Wright function $W_{\lambda,\mu}(z)$ can be represented in terms of the more familiar generalized hypergeometric functions (see [13, section 2.1]). In particular, when $\lambda = 1$ and $\mu = p+1$, the functions $W_{1,p+1}(-z^2/4)$ are expressed in terms of the Bessel functions $J_p(z)$, given as follows:

$$J_p(z) = \left(\frac{z}{2}\right)^p W_{1,p+1}\left(-\frac{z^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+p+1)} \frac{\left(\frac{z}{2}\right)^{2n+p}}{n!}.$$

Furthermore, the function $W_{\lambda,p+1}(-z) \equiv J_p^{\lambda}(z)\ (\lambda > 0, p > -1)$ is known as the generalized Bessel function (misnamed also as the Bessel-Maitland function).

2. PRELIMINARIES

Let A be the class of analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ functions $f(z)$, normalized by $f(0) = 0 = f'(0) - 1$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and S denote the class of all functions in A which are univalent in U.

Also, let $S^*(\alpha)$, $K(\alpha)$ and $C(\alpha)$ denote the subclasses of A consisting of functions which are, respectively, starlike, convex and close-to-convex with respect to starlike function $g(z)$ (need not be normalized) of order $\alpha \in [0,1)$ in the open unit disk U. Thus, we have (see for details, [14, 15], also [16])

$$S^*(\alpha) = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U \right\}, \alpha \in [0,1),$$

$$C(\alpha) = \left\{ f \in A : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in U \right\}, \alpha \in [0,1)$$

and

$$K(\alpha) = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ z \in U,\ g \in S^* \right\}, \alpha \in [0,1)$$

where, for convenience,

$$S^* = S^*(0),\ C = C(0) \text{ and } K = K(0)$$

are, respectively, starlike, convex and close-to-convex functions in U. It is well known that close-to-convex functions are univalent in U, but not necessarily the converse. It is easy to verify that $C \subset S^* \subset K$. For details on these classes, one could refer to the monograph by Goodman [15].

Recently, several researchers studied classes of analytic functions involving special functions $F \subset A$, to find different conditions such that the members of F have certain geometric properties such as starlkeness, convexity or close-to-convexity in U. There is an extensive literature dealing with geometric properties of different types of hypergeometric functions, especially for generalized Gaussian, Kummer hypergeometric and Bessel functions (see [2, 7-12]).

An interesting generalization of the function classes $S^*(\alpha)$, $C(\alpha)$ and $K(\alpha)$ are provided by the classes $S^*(\alpha,\beta)$, $C(\alpha,\beta)$ and $K(\alpha,\beta)$ of functions $f \in A$, which satisfies the following conditions:

$$S^*(\alpha,\beta)=\left\{ f \in A: \mathrm{Re}\left(\frac{zf'(z)+\beta z^2 f''(z)}{f(z)}\right) > \alpha, z \in U \right\}, \alpha,\beta \in [0,1),$$

$$C(\alpha,\beta)=\left\{ f \in A: \mathrm{Re}\left[\frac{\left(zf'(z)+\beta z^2 f''(z)\right)'}{f'(z)}\right] > \alpha,\ z \in U \right\}, \alpha,\beta \in [0,1)$$

and

$$K(\alpha,\beta)=\left\{ f \in A:\ \mathrm{Re}\left(\frac{zf'(z)+\beta z^2 f''(z)}{g(z)}\right) > \alpha,\ z \in U,\ g \in S^*(\alpha,\beta) \right\}, \alpha,\beta \in [0,1)$$

with respect to function $g(z)$ (need not be normalized), respectively.

The classes $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$ was studied by Murugusundaramoorthy *et* al. [17].

Clearly, for $\beta = 0$, we have $S^*(\alpha,0)=S^*(\alpha)$, $C(\alpha,0)=C(\alpha)$ and $K(\alpha,0)=K(\alpha)$.

Note 2.1 *The class $K(\alpha,\beta)$ is the first time considered and examined in this paper.*

Remark 2.1 *For $f \in A$, it is easy to verify that $f(z) \in C(\alpha,\beta) \Leftrightarrow zf'(z) \in S^*(\alpha,\beta)$.*

Observe that the Wright function $W_{\lambda,\mu}(z)$ defined by (1.1) does not belong to the class A. Thus, it is natural to consider the following two kinds of normalization of the Wright function:

$$W^{(1)}_{\lambda,\mu}(z)=\Gamma(\mu)zW_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(\mu)}{\Gamma\left(\lambda n+\mu\right)}\frac{z^{n+1}}{n!},\ \lambda > -1, \mu > 0, z \in U$$

and

$$W^{(2)}_{\lambda,\mu}(z)=\Gamma(\lambda+\mu)\left[W_{\lambda,\mu}(z)-\frac{1}{\Gamma(\mu)}\right]=\sum_{n=0}^{\infty}\frac{\Gamma\left(\lambda+\mu\right)}{\Gamma\left(\lambda n+\lambda+\mu\right)}\frac{z^{n+1}}{(n+1)!}, \lambda > -1, \lambda+\mu > 0, z \in U.$$

Easily, we write

$$W^{(1)}_{\lambda,\mu}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma\left(\lambda(n-1)+\mu\right)}\frac{z^n}{(n-1)!}, \lambda > -1, \mu > 0, z \in U, \tag{2.1}$$

$$W^{(2)}_{\lambda,\mu}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma\left(\lambda+\mu\right)}{\Gamma\left(\lambda(n-1)+\lambda+\mu\right)}\frac{z^n}{n!}, \lambda > -1, \lambda+\mu > 0, z \in U. \tag{2.2}$$

Furthermore, observe that $W^{(1)}_{\lambda,\mu}$ and $W^{(2)}_{\lambda,\mu}$ are satisfying the following relations:

$$\lambda z\left(W^{(1)}_{\lambda,\mu}(z)\right)'=\left(\mu-1\right)W^{(1)}_{\lambda,\mu-1}(z)+\left(\lambda-\mu+1\right)W^{(1)}_{\lambda,\mu}(z),$$

$$\lambda z\left(W^{(2)}_{\lambda,\mu}(z)\right)'=\left(\lambda+\mu-1\right)W^{(2)}_{\lambda,\mu-1}(z)+\left(1-\mu\right)W^{(2)}_{\lambda,\mu}(z),$$

$$z\left(W^{(2)}_{\lambda,\mu}(z)\right)' = W^{(1)}_{\lambda,\lambda+\mu}(z) \text{ and } V'_{\lambda,\mu}(z) = \frac{\Gamma(\mu)}{\Gamma(\lambda+\mu)} V_{\lambda,\lambda+\mu}(z)$$

where

$$V_{\lambda,\mu}(z) = \frac{W^{(1)}_{\lambda,\mu}(z)}{z}. \tag{2.3}$$

In this paper, we give sufficient conditions for the functions represented with normalized Wright functions to be in the subclasses $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$, $\alpha,\beta \in [0,1)$ of analytic functions in the open unit disk.

The following lemma will be required in our present investigation.

Lemma 2.1 [18] *A function* $f \in A$ *belongs to the class* $S^*(\alpha,\beta)$ *if*

$$\sum_{n=2}^{\infty}\left[n+\beta n(n-1)-\alpha\right]\left|a_n\right| \le 1-\alpha .$$

3. SOME INTEGRAL OPERATORS OF NORMALIZED WRIGHT FUNCTIONS

In this section, we will examine some geometric properties of the integral operators of normalized Wright functions.

Let

$$G^{(1)}_{\lambda,\mu}(z) = \int_0^z \frac{W^{(1)}_{\lambda,\mu}(t)}{t}dt \text{ and } G^{(2)}_{\lambda,\mu}(z) = \int_0^z \frac{W^{(2)}_{\lambda,\mu}(t)}{t}dt,\ z \in U, \tag{3.1}$$

where $W^{(1)}_{\lambda,\mu}(z)$ and $W^{(2)}_{\lambda,\mu}(z)$ are functions defined by (2.1) and (2.2).

From (3.1) it is easy to verify that $G^{(1)}_{\lambda,\mu}, G^{(2)}_{\lambda,\mu} \in A$.

In the following theorems, we will give sufficient conditions so that functions $G^{(1)}_{\lambda,\mu}(z)$ and $G^{(2)}_{\lambda,\mu}(z)$ are in the classes $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$, respectively.

Theorem 3.1 *Let* $\lambda \ge 1$ *and assume the condition*

$$(1-\alpha)\mu\left(\mu+2-\mu e^{\frac{1}{\mu}}\right) - \mu\left(e^{\frac{1}{\mu}}-1\right) - \beta e^{\frac{1}{\mu}} \ge 0 \tag{3.2}$$

then function $G^{(1)}_{\lambda,\mu}(z)$ *belongs to the class* $S^*(\alpha,\beta)$.

Proof. Since

$$G^{(1)}_{\lambda,\mu}(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\frac{z^n}{n!}$$

by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty}\left[n+\beta n(n-1)-\alpha\right]\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\frac{1}{n!} \le 1-\alpha.$$

Let

$$L_1(\lambda,\mu;\alpha,\beta)=\sum_{n=2}^{\infty}\left[n+\beta n(n-1)-\alpha\right]\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\frac{1}{n!}.$$

By simple computation, we write

$$L_1(\lambda,\mu;\alpha,\beta)=\sum_{n=2}^{\infty}\frac{\beta}{(n-2)!}\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}+$$
$$\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)\frac{1}{(n-1)!}\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}+\sum_{n=2}^{\infty}\frac{1-\alpha}{n!}\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}.$$

Under the hypothesis $\lambda \geq 1$, the inequality $\Gamma(n-1+\mu)\leq\Gamma((n-1)\lambda+\mu)$ for $n\in\mathbb{N}$, holds, which is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\leq\frac{1}{(\mu)_{n-1}},\ n\in\mathbb{N} \tag{3.3}$$

where

$$(\mu)_n=\frac{\Gamma(n+\mu)}{\Gamma(\mu)}=\mu(\mu+1)(\mu+2)\cdots(\mu+n-1),\ (\mu)_0=1$$

is Pochhammer (or Appell) symbol, defined in terms of Euler gamma function.

Using (3.3), we obtain

$$L_1(\lambda,\mu;\alpha,\beta)\leq\sum_{n=2}^{\infty}\frac{\beta}{(n-2)!}\frac{1}{(\mu)_{n-1}}+\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\frac{1}{(\mu)_{n-1}}+\sum_{n=2}^{\infty}\frac{(1-\alpha)}{n!}\frac{1}{(\mu)_{n-1}}.$$

Further, the inequality

$$(\mu)_{n-1}=\mu(\mu+1)(\mu+2)\cdots(\mu+n-2)\geq\mu^{n-1},\ n\in\mathbb{N} \tag{3.4}$$

holds, which is equivalent to $1/(\mu)_{n-1}\leq 1/\mu^{n-1},\ n\in\mathbb{N}.$

Using inequality (3.4), we obtain

$$L_1(\lambda,\mu;\alpha,\beta)\leq\sum_{n=2}^{\infty}\frac{\beta}{(n-2)!}\frac{1}{\mu^{n-1}}+\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\frac{1}{\mu^{n-1}}+\sum_{n=2}^{\infty}\frac{1-\alpha}{n!}\frac{1}{\mu^{n-1}}$$
$$=\frac{\beta}{\mu}e^{\frac{1}{\mu}}+\left(e^{\frac{1}{\mu}}-1\right)+(1-\alpha)\mu\left(e^{\frac{1}{\mu}}-\frac{\mu+1}{\mu}\right)\leq 1-\alpha,$$

which is equivalent to

$$(1-\alpha)\mu\left(\mu+2-\mu e^{\frac{1}{\mu}}\right)-\mu\left(e^{\frac{1}{\mu}}-1\right)-\beta e^{\frac{1}{\mu}}\geq 0.$$

Thus, the proof of Theorem 3.1 is complete.

By setting $\beta=0$ in Theorem 3.1 and using the relationship $S^*(\alpha,0)=S^*(\alpha)$, we arrive at the following corollary.

Corollary 3.1 *The function $G^{(1)}_{\lambda,\mu}(z)$ belongs to the class $S^*(\alpha)$ if $\lambda\geq 1$ and the following condition is satisfied:*

$$(1-\alpha)\left(\mu+2-\mu e^{\frac{1}{\mu}}\right)-\left(e^{\frac{1}{\mu}}-1\right)\geq 0.$$

By taking $\alpha=0$ in Corollary 3.1 and using relationship $S^*(0)=S^*$, we arrive at the following corollary.

Corollary 3.2 *The function* $G_{\lambda,\mu}^{(1)}(z)$ *belongs to the class* S^* *if* $\lambda\geq 1$ *and* $\mu>x_1$. *Here,* $x_1\approx 1.9133$ *is the root of the equation*

$$x+3-(x+1)e^{\frac{1}{x}}=0.$$

Proof. Let $\varphi(x)=x+3-(x+1)e^{1/x},\ x>0$. By simple computation, we obtain

$$\varphi'(x)=1-\frac{x^2-x-1}{x^2}e^{\frac{1}{x}},\ x>0.$$

We observe that $\varphi'(x)>0$ for each $x>0$. Thus, function $\varphi(x)$ is an increasing function.

Hence, $\mu+3-(\mu+1)e^{\frac{1}{\mu}}>0$ for every $\mu>x_1$ where $x_1\approx 1.9133$ is the root of the equation

$$x+3-(x+1)e^{\frac{1}{x}}=0.$$

Thus, the proof of Corollary 3.2 is complete.

Theorem 3.2 *Let* $\lambda\geq 1$ *and assume the condition*

$$(1-\alpha)\left(2-e^{\frac{1}{\mu}}\right)\mu^2-\left[\mu+(2\mu+1)\beta\right]e^{\frac{1}{\mu}}\geq 0,$$

then the function $G_{\lambda,\mu}^{(1)}(z)$ *belongs to the class* $C(\alpha,\beta)$.

Proof. The function $G_{\lambda,\mu}^{(1)}(z)$ belongs to the class $C(\alpha,\beta)$ if and only if $z\cdot(G_{\lambda,\mu}^{(1)}(z))'\in S^*(\alpha,\beta)$.

Since

$$z\cdot\left(G_{\lambda,\mu}^{(1)}(z)\right)'=W_{\lambda,\mu}^{(1)}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\frac{z^n}{(n-1)!}$$

by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty}\left[n+\beta n(n-1)-\alpha\right]\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\frac{1}{(n-1)!}\leq 1-\alpha,$$

or

$$\sum_{n=2}^{\infty}\left[n^2\beta+n(1-\beta)-\alpha\right]\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\frac{1}{(n-1)!}\leq 1-\alpha.$$

Let

$$L_2(\lambda,\mu;\alpha,\beta)=\sum_{n=2}^{\infty}\left[n^2\beta+n(1-\beta)-\alpha\right]\frac{\Gamma(\mu)}{\Gamma\left((n-1)\lambda+\mu\right)}\frac{1}{(n-1)!}.$$

Setting $n^2=(n-1)(n-2)+3(n-1)+1,\ n=(n-1)+1$ and by simple computation, we get

$$L_2(\lambda,\mu;\alpha,\beta)=\sum_{n=3}^{\infty}\frac{\beta}{(n-3)!}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}+$$
$$\sum_{n=2}^{\infty}\frac{1+2\beta}{(n-2)!}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}+\sum_{n=2}^{\infty}\frac{1-\alpha}{(n-1)!}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}.$$

Using (3.3) and (3.4), we obtain

$$L_2(\lambda,\mu;\alpha,\beta)\le\sum_{n=3}^{\infty}\frac{\beta}{(n-3)!}\frac{1}{\mu^{n-1}}+\sum_{n=2}^{\infty}\frac{1+2\beta}{(n-2)!}\frac{1}{\mu^{n-1}}+\sum_{n=2}^{\infty}\frac{1-\alpha}{(n-1)!}\frac{1}{\mu^{n-1}}=$$
$$\frac{\beta}{\mu^2}e^{\frac{1}{\mu}}+\frac{1+2\beta}{\mu}e^{\frac{1}{\mu}}+(1-\alpha)\left(e^{\frac{1}{\mu}}-1\right)\le 1-\alpha,$$

which is equivalent to

$$(1-\alpha)\left(2-e^{\frac{1}{\mu}}\right)\mu^2-\left(\mu+(2\mu+1)\beta\right)e^{\frac{1}{\mu}}\ge 0.$$

Thus, the proof of Theorem 3.2 is complete.

By setting $\beta=0$ in Theorem 3.2 and using the relationship $C(\alpha,0)=C(\alpha)$, we arrive at the following corollary.

Corollary 3.3 *The function $G_{\lambda,\mu}^{(1)}(z)$ belongs to the class $C(\alpha)$ if $\lambda\ge 1$ and the following condition is satisfied:*

$$(1-\alpha)\left(2-e^{\frac{1}{\mu}}\right)\mu-e^{\frac{1}{\mu}}\ge 0.$$

By taking $\alpha=0$ in Corollary 3.3 and using relationship $C(0)=C$, we arrive at the following corollary.

Corollary 3.4 *The function $G_{\lambda,\mu}^{(1)}(z)$ belongs to the class C if $\lambda\ge 1$ and $\mu>x_2$. Here, $x_2\approx 2.6679$ is the root of the equation*

$$2x-(x+1)e^{\frac{1}{x}}=0.$$

Proof. Let $\phi(x)=2x-(x+1)e^{1/x},\ x>0$. By simple computation, we obtain

$$\phi'(x)=2-\frac{x^2-x-1}{x^2}e^{\frac{1}{x}},\ x>0.$$

We observe that $\phi'(x)>0$ for each $x>0$. Thus, function $\phi(x)$ is an increasing function.

Hence, $2\mu-(\mu+1)e^{\frac{1}{\mu}}>0$ for every $\mu>x_2$ where $x_2\approx 2.6679$ is the root of the equation

$$2x-(x+1)e^{\frac{1}{x}}=0.$$

Thus, the proof of Corollary 3.4 is complete.

Theorem 3.3 *Let $\lambda\ge 1$ and assume the condition*

$$(1-\alpha)\left[2-(\lambda+\mu)(e^{\frac{1}{\lambda+\mu}}-1)\right]-(\lambda+\mu)\left(e^{\frac{1}{\lambda+\mu}}-1\right)+\beta\left[(\lambda+\mu-1)(e^{\frac{1}{\lambda+\mu}}-1)-1\right]+1\geq 0,$$

then function $G^{(2)}_{\lambda,\mu}(z)$ *belongs to the class* $S^{*}(\alpha,\beta)$.

Proof. Since

$$G^{(2)}_{\lambda,\mu}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma((n-1)\lambda+\lambda+\mu)}\frac{z^{n}}{n!n}$$

by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty}[n+\beta n(n-1)-\alpha]\frac{\Gamma(\lambda+\mu)}{\Gamma((n-1)\lambda+\lambda+\mu)}\frac{1}{n!n}\leq 1-\alpha.$$

Let

$$L_{3}(\lambda,\mu;\alpha,\beta)=\sum_{n=2}^{\infty}[n+\beta n(n-1)-\alpha]\frac{\Gamma(\lambda+\mu)}{\Gamma((n-1)\lambda+\lambda+\mu)}\frac{1}{n!n}.$$

Simple computation, we write

$$L_{3}(\lambda,\mu;\alpha,\beta)=\sum_{n=2}^{\infty}\frac{\beta}{(n-1)!}\frac{\Gamma(\lambda+\mu)}{\Gamma((n-1)\lambda+\lambda+\mu)}+$$
$$\sum_{n=2}^{\infty}\left(1-\beta-\frac{1}{n}\right)\frac{1}{n!}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}+\sum_{n=2}^{\infty}\frac{1-\alpha}{n!n}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}.$$

Using inequality (3.3) and (3.4), with $\mu\equiv\lambda+\mu$, we obtain

$$L_{3}(\lambda,\mu;\alpha,\beta)\leq\sum_{n=2}^{\infty}\frac{\beta}{(n-1)!}\frac{1}{(\lambda+\mu)^{n-1}}+\sum_{n=2}^{\infty}\frac{2-\alpha-\beta}{n!}\frac{1}{(\lambda+\mu)^{n-1}}=\beta\left(e^{\frac{1}{\lambda+\mu}}-1\right)+$$
$$(2-\alpha-\beta)(\lambda+\mu)\left(e^{\frac{1}{\lambda+\mu}}-\frac{\lambda+\mu+1}{\lambda+\mu}\right)\leq 1-\alpha,$$

which is equivalent to

$$(1-\alpha)\left[2-(\lambda+\mu)\left(e^{\frac{1}{\lambda+\mu}}-1\right)\right]-(\lambda+\mu)\left(e^{\frac{1}{\lambda+\mu}}-1\right)+\beta\left[(\lambda+\mu-1)\left(e^{\frac{1}{\lambda+\mu}}-1\right)-1\right]+1\geq 0.$$

Thus, the proof of Theorem 3.3 is complete.

By setting $\beta=0$ in Theorem 3.3 and using the relationship $S^{*}(\alpha,0)=S^{*}(\alpha)$, we arrive at the following corollary.

Corollary 3.5 *The function* $G^{(2)}_{\lambda,\mu}(z)$ *belongs to the class* $S^{*}(\alpha)$ *if* $\lambda\geq 1$ *and the following condition is satisfied:*

$$(1-\alpha)\left[2-(\lambda+\mu)\left(e^{\frac{1}{\lambda+\mu}}-1\right)\right]-(\lambda+\mu)\left(e^{\frac{1}{\lambda+\mu}}-1\right)+1\geq 0.$$

By taking $\alpha=0$ in Corollary 3.5 and using relationship $S^{*}(0)=S^{*}$, we arrive at the following corollary.

Corollary 3.6 *The function* $G^{(2)}_{\lambda,\mu}(z)$ *belongs to the class* S^* *if* $\lambda \geq 1$ *and* $\lambda+\mu > x_3$. *Here,* $x_3 \approx 1.3112$ *is the root of the equation*

$$3-2x\left(e^{\frac{1}{x}}-1\right)=0.$$

Proof. Let $\psi(x)=2x+3-2xe^{1/x},\ x>0$. By simple computation, we obtain

$$\psi'(x)=2-\frac{2(x-1)}{x}e^{\frac{1}{x}},\ x>0.$$

We observe that $\psi'(x)>0$ for each $x>0$. Thus, function $\psi(x)$ is an increasing function.

Hence, $3-2(\lambda+\mu)(e^{\frac{1}{\lambda+\mu}}-1)>0$ for every $\lambda+\mu>x_3$ where $x_3 \approx 1.3112$ is the root of the equation $3-2x(e^{\frac{1}{x}}-1)=0$.

Thus, the proof of Corollary 3.6 is complete.

Theorem 3.4 *Let* $\lambda \geq 1$ *and assume the condition*

$$(1-\alpha)(\lambda+\mu)\left[\lambda+\mu+2-(\lambda+\mu)e^{\frac{1}{\lambda+\mu}}\right]-(\lambda+\mu)\left(e^{\frac{1}{\lambda+\mu}}-1\right)-\beta e^{\frac{1}{\lambda+\mu}} \geq 0, \tag{3.5}$$

then the function $G^{(2)}_{\lambda,\mu}(z)$ *belongs to the class* $C(\alpha,\beta)$.

Proof. The function $G^{(2)}_{\lambda,\mu}(z)$ belongs to the class $C(\alpha,\beta)$ if and only if $z\cdot(G^{(2)}_{\lambda,\mu}(z))' \in S^*(\alpha,\beta)$.

Since

$$z\cdot\left(G^{(2)}_{\lambda,\mu}(z)\right)'=W^{(2)}_{\lambda,\mu}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{z^n}{n!}$$

by virtue of Lemma 2.1, if suffices to show that

$$\sum_{n=2}^{\infty}\left[n+\beta n(n-1)-\alpha\right]\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!}\leq 1-\alpha.$$

Since

$$L_1(\lambda,\lambda+\mu;\alpha,\beta)=\sum_{n=2}^{\infty}\left[n+\beta n(n-1)-\alpha\right]\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!}$$

the proof of Theorem 3.4 is clear from the assume of Theorem 3.1.

Indeed, we obtain condition (3.5) if in (3.2) replace μ with $\lambda+\mu$. This evidently completes the proof.

By setting $\beta=0$ in Theorem 3.4 and using the relationship $C(\alpha,0)=C(\alpha)$, we arrive at the following corollary.

Corollary 3.7 *The function* $G^{(2)}_{\lambda,\mu}(z)$ *belongs to the class* $C(\alpha)$ *if* $\lambda \geq 1$ *and the following condition is satisfied:*

$$(1-\alpha)\left[\lambda+\mu+2-(\lambda+\mu)e^{\frac{1}{\lambda+\mu}}\right]-\left(e^{\frac{1}{\lambda+\mu}}-1\right)\geq 0.$$

By taking $\alpha = 0$ in Corollary 3.7 and using relationship $C(0) = C$, we arrive at the following corollary.

Corollary 3.8 *The function* $G^{(2)}_{\lambda,\mu}(z)$ *belongs to the class* C *if* $\lambda \geq 1$ *and* $\lambda + \mu > x_1$. *Here,* $x_1 \approx 1.9133$ *is the root of the equation*

$$x + 3 - (x+1)e^{\frac{1}{x}} = 0 .$$

4. CONCLUDING REMARKS

In this study, two kinds of integral operators o of normalization Wright function are investigated. These integral operators are examined in the generalized classes of the starlike, convex and close-to-convex function classes, respectively.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

REFERENCES

[1] Wright, E.M., “On the coefficients of power series having exponential singularities”, ***J. London Math. Soc.*** 8: 71-79 (1933).

[2] Podlubny, I., “Fractional differential equations”, San Diego, ***Academic Press***, (1999).

[3] Samko, S.G., Kilbas, A.A., Marichev,.I., “Fractional integrals and derivatives: theory and Applications”, Gordon and Breach, ***New York***, (1993).

[4] Mainardi, F., “Fractional calculus: some basic problems in continuum and statistical Mechanics”, In: Fractals and Fractional Calculus in Continuum Mechanics (Eds. A. Carpinteri and F. Mainardi), ***Springer Verlag***, Wien, 291-348 (1971).

[5] Buckwar, E., Luchko, Yu.,“Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations”, ***J. Math. Appl.*** 227: 81-97 (1998).

[6] Luchko, Yu., Gorenflo, R., “Scale-invariant solutions of a partial differential equation of fractional order”, ***Fractional Calculus & Applied Analysis.*** 1(1): 63-78 (1998).

[7] Baricz, A., Ponnusamy, S.,“Starlikeness and convexity of generalized Bessel functions”, ***Integral Transform. Spec. Funct.*** 21(9): 641-653 (2010).

[8] Miller, S.S., Mocanu, P.T.,“Univalence of Gaussian and Confluent hypergeometric functions”, ***Proc. Amer. Math. Soc.*** 110(2): (1990).

[9] Ponnusamy S., Ronning, F, “Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a half plane”, ***Integral Transform. Spec.*** Funct. 8: 121-138 (1999).

[10] Ponnusamy, S., Singh, V., Vasundhra,P., “Starlikeness and convexity of an integral Transform”, ***Integral Transform. Spec. Funct.*** 15(3): 267-280 (2004).

[11] Ponnusamy, S., Vuorinen, M., “Univalence and convexity properties for Gaussian hypergeometric functions”, ***Rocky Mountain J. Math.*** 31(1): 327-353 (2001).

[12] Ponnusamy, S., Vuorinen, M., “Univalence and convexity properties for confluent hypergeometric functions”, ***Complex Variable Theory Appl.*** 36(1): 73-97 (1998).

[13] Gorenflo, R., Luchko, Yu., Mainardi, F., "Analytic properties and applications of Wright Functions", ***Frac Cal Appl Anal.*** 2(4): 383-414 (1999).

[15] Duren, P.L., "Univalent Functions", Grundlehren der Mathematischen Wissenshaften, Bd. 259, ***Springer-Verlag***, New York, (1983).

[15] Goodman A.W., "Univalent Functions", Volume I, Polygonal, ***Washington***, (1983).

[16] Srivastava, H.M., Owa, S., Editors, "Current Topics in Analytic Function Theory", ***World Scientific***, Singapore, (1992).

[17] Murugusundaramoorthy, G., Vijaya, K., Porwal, S., "Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series", ***Hacettepe Journal of Mathematics and Statistics,*** DOI: 10.15672/HJMS201664513110.

[18] Thulasiram, T., Suchithra, K, Sudharsan, T.V., Murugusundaramoorthy, G., "Some inclusion results associated with certain subclass of analytic functions involving Hohlov operator", ***Rev. R. Acad. Cienc. Exactas, Fis. Nat.Ser.-A Mat.*** 108 (2): 711-720 (2014).