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Automorphisms with Lie Ideals and (η,ξ)-derivations

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Abstract
In this study, prime ring S, which has been accepted to have charS≠2 characteristic, has been supposed also to have automorphism of η, ξ, κ, ε, λ and μ. It was aimed to examine the derivatives and automorphisms that provide certain identities on the Lie ideals of prime rings.

1. Introduction

Let S be a ring, η, ξ, κ, ε, λ and μ are mappings of S, and Z(S) denotes the center of S. By a prime ring, we mean a ring S in which for every x,y∈S, xSy=(0) implies x=0 or y=0. The operation of xy-yx, represented by the [x, y] commutator, and the operation of xy+yx, represented by the (x, y) commutator, has been selected from the elements of S. Morever [x, y]η,ξ=xη(y)-ξ(y)x and (x, y])η,ξ=xη(y)+ξ(y)x for x,y∈S. For subsets M, N of S, let [M, N]η,ξ ={[x, y]η,ξ |x∈M and y∈N}.

The set K that satisfies the [S, K]⊂K property is called a Lie ideal of S. Here, the set K is an additive subset of the ring S.

If $S=\left\{\begin{pmatrix} x & y \\ z & t \end{pmatrix} \middle| x,y,z,t\in\mathbb{Z}\right\}$ and $K=\left\{\begin{pmatrix} x & y \\ z & x \end{pmatrix} \middle| x,y,z\in\mathbb{Z}\right\}$ then K is a Lie ideal of S.

For all x,y∈S when both conditions F(x+y)=F(x)+F(y) and F(xy)=F(x)y+xF(y) is satisfied, F mapping is called as a derivation of S. For all x,y∈S, when the condition of d(xy)=d(x)η (y)+ ξ(x)d(y), d mapping is called as a (η,ξ)-derivation of S.

Over the years, the commutativity of the rings has been investigated under certain derivative conditions. Various generalizations have been made in examining the commutativity of a ring under these conditions.On the other hand, these conditions, which are considered as d ordinary derivations, are examined in α - derivation, (σ,τ)-derivation.

On the other hand, the commutativity of the ring was investigated by taking its ideal and (or) Lie ideal instead of the R ring in these conditions. In 1970, I. N. Herstein verified the R ring is commutative if the condition [d(S), d(S)]⊆ Z(S) is met, where S is a prime ring and its characteristic is different from the value of 2. d is a derivative of the S ring and is nonzero [5]. In the same work, I.N. Herstein also proved that the S ring is commutative if the d(S) condition is satisfied. Later, J. Bergen et al. proved that K⊂Z(S) when d(K)⊂Z(S) and K is the Lie ideal of S that has a value other than 2 and S is a ring that is prime [6].

In this study, prime ring S, which has been accepted to have charS≠2 characteristic, has been supposed also to have automorphism of η, ξ, κ, ε, λ and μ. Z(S) will denote the centre of an associative ring and Cη,ξ={c∈S|cη(r)=ξ(r)c, ∀ r∈S}. The equations frequently used in this study have been given below:

[xy,z]η,ξ=x[y,z]η,ξ-[x,ξ(z)]y=x[y,η(z)]+[x, z]η,ξy

[x,yz]η,ξ=ξ(y)[x, z]η,ξ+[x, y]η,ξη(z)

(xy,z)η,ξ=x(y,z)η,ξ-[x,ξ(z)]y=x[y,η(z)]+(x, z)η,ξy

(x,yz)η,ξ=ξ(y)(x, z)η,ξ+[x, y]η,ξη(z)=ξ(y)[x, z]η,ξ+(x, y)η,ξη(z)

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2. Preliminary

Lemma 1 [1] If the condition $[[J, a]_{\eta,\xi}, b]_{\kappa,\varepsilon}=0$ is provided for the nonzero J ideal of the S ring, then $[\xi(a), \varepsilon(b)]$ becomes zero.

Lemma 2 [1] If the condition $[[a, J]_{\eta,\xi},b]_{\kappa,\varepsilon}=0$ is provided for the nonzero J ideal of the S ring, then $[a, \xi^{-1}\varepsilon(b)]_{\eta,\xi}$ becomes zero or $b\in Z(S)$.

Lemma 3 [2] If the condition $d(K)\subset C_{\kappa,\varepsilon}(S)$ is provided for the nonzero Lie ideal K of the S ring and d is a nonzero (η, ξ)-derivation of S, then $K\subset Z(S)$.

Lemma 4 [2] If the condition $d(K)\subset C_{\kappa,\varepsilon}(S)$ is provided for the nonzero Lie ideal K of the S ring then $a\in C_{\kappa,\varepsilon}(S)$ or $K\subset Z(S)$.

Lemma 5 [2] Let $d:S\to S$ be a nonzero (η, ξ)-derivation and $a\in S$. If J is an ideal of S that is nonzero such that $[d(J), a]_{\kappa,\varepsilon}\subset C_{\kappa,\varepsilon}(S)$ then $a\in Z(S)$ or $\{d\xi^{-1}\varepsilon(a)=0$ and $d\eta^{-1}\kappa(a)=0\}$.

Lemma 6 [2] If the condition $[a, d(J)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ is provided for the nonzero ideal J of the S ring and d is a nonzero (η, ξ)-derivation of S and also $d\eta=\eta d$, $d\xi=\xi d$ then $a\in C_{\kappa,\varepsilon}(S)$ or S is commutative.

3. Automorphisms with Lie İdeals

Let $T:S\to S$ be a nonzero automorphism and $d:S\to S$ a nonzero (η, ξ)-derivation. Let K be a nonzero Lie ideal of S. In this note, some results on automorphisms and Lie ideals of prime rings have been proven as seen below.

Lemma 7 Let T be an automorphism of S and $a, b\in S$.
(i) $a\in C_{\kappa T,\varepsilon T}(S)$ if and only if $a\in C_{\kappa,\varepsilon}(S)$,
(ii) $T(a)\in C_{\kappa,\varepsilon}(S) \Leftrightarrow a\in C_{T^{-1}\kappa,T^{-1}\varepsilon}(S)$,
(iii) $[T(a),b]_{\kappa,\varepsilon}\in C_{\lambda,\mu}(S) \Leftrightarrow [a, b]_{T^{-1}\kappa,T^{-1}\varepsilon}\in C_{T^{-1}\lambda,T^{-1}\mu}(S)$,
(iv) If $C_{\lambda,\mu}(S)=0$ then $C_{T\lambda,T\mu}(S)=0$,
(v) $(T(a), b)_{\kappa,\varepsilon}\in C_{\lambda,\mu}(S) \Leftrightarrow (a, b)_{T^{-1}\kappa,T^{-1}\varepsilon}\in C_{T^{-1}\lambda,T^{-1}\mu}(S)$.

Proof

(i) $a\in C_{\kappa T,\varepsilon T}(S)\Leftrightarrow a\kappa T(r)-\varepsilon T(r)a=0,\forall r\in S\Leftrightarrow[a, T(S)]_{\kappa,\varepsilon}=0 \Leftrightarrow [a, S]_{\kappa,\varepsilon}=0 \Leftrightarrow a\in C_{\kappa,\varepsilon}(S)$.
(ii) $T(a)\in C_{\kappa,\varepsilon}(S \Leftrightarrow T(a)\kappa(r)-\varepsilon(r)T(a)=0,\forall r\in S$
$\Leftrightarrow T(aT^{-1}\kappa(r)-T^{-1}\varepsilon(r)a)=0 ,\forall r\in S$
$\Leftrightarrow aT^{-1}\kappa(r)-T^{-1}\varepsilon(r)a=0,\forall r\in S$
$\Leftrightarrow a\in C_{T^{-1}\kappa,T^{-1}\varepsilon}(S)$
(iii) $T(a)\kappa(b)- \varepsilon(b)T(a)\in C_{\lambda,\mu}(S)$
$\Leftrightarrow T(aT^{-1}\kappa(b)-T^{-1}\varepsilon(b)a)\in C_{\lambda,\mu}(S)$
$\Leftrightarrow (aT^{-1}\kappa(b)-T^{-1}\varepsilon(b)a)\in C_{T^{-1}\lambda,T^{-1}\mu}(S)$ by (ii)
$\Leftrightarrow[a, b]_{T^{-1}\kappa, T^{-1}\varepsilon}\in C_{T^{-1}\lambda, T^{-1}\mu}(S)$.
(iv) Let $C_{\lambda,\mu}(S)=0$. Then for any $a\in C_{T\lambda T,\mu}(S)$, $aT\lambda(r)-T\mu(r)a=0,\forall r\in S$
$\Rightarrow T^{-1}(a)\lambda(r)-\mu(r)T^{-1}(a)=0,\forall r\in S$
$\Rightarrow [T^{-1}(a), S]_{\lambda,\mu}=0 \Rightarrow T^{-1}(a)\in C_{\lambda,\mu}(S)=0$
$\Rightarrow T^{-1}(a)=0$
$\Rightarrow a=0$
$\Rightarrow C_{T\lambda,T\mu}(S)=0$.
On the other hand let $C_{T\lambda T,\mu}(S)=0$. Then we get $b\in C_{\lambda,\mu}(S)$
$\Rightarrow b\lambda(r)-\mu(r)b=0,\forall r\in S$
$\Rightarrow T(b)T\lambda(r)-T\mu(r)T(b))=0,\forall r\in S$
$\Rightarrow T(b)\in C_{T\lambda,T\mu}(S)=0$
$\Rightarrow T(b)=0$
$\Rightarrow b=0$
$\Rightarrow C_{\lambda,\mu}(S)=0$
(v) $(T(a), b)_{\kappa,\varepsilon}\in C_{\lambda,\mu}(S)$
$\Leftrightarrow T(a)\kappa(b)+\varepsilon(b)T(a)\in C_{\lambda,\mu}(S)$
$\Leftrightarrow T(aT^{-1}\kappa(b)+T^{-1}\varepsilon(b)a)\in C_{\lambda,\mu}(S)$
$\Leftrightarrow (aT^{-1}\kappa(b)+T^{-1}\varepsilon(b)a)\in C_{T^{-1}\lambda,T^{-1}\mu}(S)$ by (ii)
$\Leftrightarrow (a, b)_{T^{-1}\kappa,T^{-1}\varepsilon}\in C_{T^{-1}\lambda,T^{-1}\mu}(S)$.

Lemma 8 Let $T:S\to S$ be a ring automorphism and $a, b, c\in S$. The equations that have been given below are true.
(i) $T[a, b]_{\kappa,\varepsilon}=[T(a), b]_{T\kappa,T\varepsilon,}$
(ii) $T(C_{\kappa,\varepsilon}(S))=C_{T\kappa,T\varepsilon}(S)$,
(iii) $[a, T(b)]_{\kappa,\varepsilon}=[a, b]_{\kappa T,\varepsilon T,}$
(iv) $(a, T(b))_{\kappa,\varepsilon}=(a, b)_{\kappa T,\varepsilon T,}$
(v) $(T(a), b)_{\kappa,\varepsilon}=T(a, b)_{T^{-1}\kappa,T^{-1}\varepsilon.}$

Proof

(i) $T[a, b]_{\kappa,\varepsilon}=T(a\kappa(b)-\varepsilon(b)a)=T(a)T\kappa(b)-T\varepsilon(b)T(a)=[T(a), b]_{T\kappa,T\varepsilon.}$
(ii) $x\in T(C_{\kappa,\varepsilon}(S))$
$\Rightarrow x=T(a), a\in C_{\kappa,\varepsilon}(S)$
$\Rightarrow T^{-1}(x)=a, a\in C_{\kappa,\varepsilon}(S)$
$\Rightarrow T^{-1}(x)\in C_{\kappa,\varepsilon}(S)$
If we use (iii), $x\in C_{T\kappa,T\varepsilon}(S)$. On the other hand if $y\in C_{T\kappa,T\varepsilon}(S)$ then $yT\kappa(r)-T\varepsilon(r)y=0,\forall r\in S$. Since T is an automorphism then we get $T^{-1}(y)\kappa(r)-\varepsilon(r)T^{-1}(y)=0,\forall r\in S$. That is $T^{-1}(y)\in C_{\kappa,\varepsilon}(S)$. This implies that $y\in T(C_{\kappa,\varepsilon}(S))$.
(iii) $[a, T(b)]_{\kappa,\varepsilon}=a\kappa T(b)- \varepsilon T(b)a=[a, b]_{\kappa T,\varepsilon T}$
(iv) $(a, T(b))_{\kappa,\varepsilon}=a\kappa T(b)+\varepsilon T(b)a=(a, b)_{\kappa T,\varepsilon T}$
(v) $(T(a), b)_{\kappa,\varepsilon}=T(a)\kappa(b)+\varepsilon(b)T(a)=T(aT^{-1}\kappa(b)+T^{-1}\varepsilon(b)a)=T(a, b)_{T^{-1}\kappa,T^{-1}\varepsilon}$

Theorem 1 Let T be an automorphism of a ring S. Let K be a Lie ideal of S.
(i) If $T(K)\subset C_{\kappa,\varepsilon}$ (S) then $K\subset Z(S)$.
(ii) If $[T(K), a]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then $a\in Z(S)$ or $K\subset Z(S)$.
(iii) If $[a, T(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then $a\in C_{\kappa,\varepsilon}(S)$ or $K\subset Z(S)$.
(iv) If $(T(K), a)_{\kappa,\varepsilon}=0$ then $a\in Z(S)$ or $K\subset Z(S)$.
(v) If $(a, T(K))_{\kappa,\varepsilon}=0$ then $a\in Z(S)$ or $K\subset Z(S)$.

Proof
(i) Let $T(K)\subset C_{\kappa,\varepsilon}(S)$.Then $K\subset C_{\lambda,\mu}(S)$ is obtained by Lemma 7 (ii), where $\lambda=T^{-1}\kappa$ and $\mu=T^{-1}\varepsilon$. That is $[K, S]_{\lambda,\mu}=0$.Here $T^{-1}\kappa$ and $T^{-1}\varepsilon$ are automorphisms of S and $[[S, K], S]_{\lambda,\mu}=0$. If we use Lemma 1 then we obtain that $[K, \varepsilon(S)]=0$ and so $K\subset Z(S)$.
(ii) Let K is a Lie ideal of R and $[T(K), a]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$. Then we have $[K, a]_{T^{-1}\kappa,\ T^{-1}\varepsilon}\subset C_{T^{-1}\lambda,T^{-1}\mu}(S)$ by Lemma 7 (iii). This gives that $[[S, K], a]_{\phi,\theta}\subset C_{\lambda,\mu}(S)$, where $\phi=T^{-1}\kappa$ and $\theta=T^{-1}\varepsilon$ and so $[d(S), a]_{\phi,\theta}\subset C_{\lambda,\mu}(S)$ where $d(r)=[v, r]$, $\forall r\in S$ is a derivation. If we use Lemma 5 then we obtain that $a\in Z(R)$ or $d\phi^{-1}\mu(a)=0$. That is $a\in Z(S)$ or $[v, \phi^{-1}\mu(a)]=0$. Considering same argument for all $v\in K$ we get

$a\in Z(S)$ or $[K,\phi^{-1}\mu(a)]=0$.

If $[K, \phi^{-1}\mu(a)]=0$ then $a\in Z(S)$ or $K\subset Z(S)$ by [1].
(iii) If $[a,T(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then we have $[a, K]_{\kappa,\varepsilon}\subset C_{\lambda T,\mu T}(S)$ by Lemma 8 (iii). This means that $a\in C_{\kappa,\varepsilon}$ (S) or $K\subset Z(S)$ by Lemma 4.
(iv) Let $(T(K), a)_{\kappa,\varepsilon}=0$. Then we have $(K, a)_{T^{-1}\kappa,T^{-1}\varepsilon}=0$ by Lemma 8 (v). This means that $a\in Z(S)$ or $K\subset Z(S)$ by Corollary 2.
(v) $(a,T(K))_{\kappa,\varepsilon}=0$ implies that $(a, K)_{\kappa T,\varepsilon T}=0$ by Lemma 8 (iv). Considering Corollary 2 we obtain that $a\in C_{\kappa T,\varepsilon T}(S)$ or $K\subset Z(S)$. If $a\in C_{\kappa T,\varepsilon T}(S)$ then $[a, K]_{\kappa T,\varepsilon T}=0$ and so $a\in C_{\kappa,\varepsilon}(S)$ or $K\subset Z(S)$ by Lemma 4.

Corollary 1 Let $T:S\rightarrow S$ be a nonzero automorphism and K, L nonzero Lie ideals of S.
(i) If $[T(K), K]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then $K\subset Z(S)$.
(ii) If $[K, T(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then $K\subset C_{\kappa,\varepsilon}(S)$ or $K\subset Z(S)$.
(iii) If $[K, L]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then $K\subset Z(S)$ or $L\subset Z(S)$.

Proof
(i) Let $[T(K), K]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$. Then we have $T(K)\subset C_{\kappa,\varepsilon}$ or $K\subset Z(S)$ by Lemma 4. If $T(K)\subset C_{\kappa,\varepsilon}(S)$ then $K\subset Z(S)$ by Theorem 1 (i).
(ii) $[K, T(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)\Rightarrow K\subset C_{\kappa,\varepsilon}(S)$ or $K\subset Z(S)$ by Theorem 1(iii). If $K\subset C_{\kappa,\varepsilon}(S)$ then $[[S, K], S]_{\kappa,\varepsilon}=0$. This implies that $[K, \xi^{-1}\varepsilon(S)]=0$ by Lemma 1. That is $K\subset Z(S)$.
(iii) If we use Lemma 4 then we have $K\subset C_{\kappa,\varepsilon}(S)$ or $L\subset Z(S)$. Considering in the proof of the (ii) we get $K\subset Z(S)$ or $L\subset Z(S)$.

Theorem 2 Let K be a nonzero Lie ideal of S. Let $T:S\rightarrow S$ be a nonzero automorphism and $d:S\rightarrow S$ a nonzero (η, ξ)-derivation.
(i) If $[T(K), d(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then $K\subset Z(S)$.
(ii) If $[d(K), T(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$ then $K\subset Z(S)$.
(iii) If $(T(K), d(K))_{\kappa,\varepsilon}=0$ then $K\subset Z(S)$.
(iv) If $(d(K), T(K))_{\kappa,\varepsilon}=0$ then $K\subset Z(S)$.

Proof
(i) Let $[T(K), d(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$. Then we have $d(K)\subset Z(S)$ or $K\subset Z(S)$ by Theorem 1 (ii). Considering [4] we obtain that, $K\subset Z(S)$ finally.
(ii) Let $[d(K), T(K)]_{\kappa,\varepsilon}\subset C_{\lambda,\mu}(S)$. Using Theorem 1 (iii), we clearly see $K\subset Z(S)$.
(iii) Let $(T(K), d(K))_{\kappa,\varepsilon}=0$. In this case, we get $d(K)\subset Z(S)$ or $K\subset Z(S)$ by Theorem 1 (ii). Considering [4] we obtain that, $K\subset Z(S)$.
(iv) Suppose that $(d(K), T(K))_{\kappa,\varepsilon}=0$. Then we have $K\subset Z(S)$ by Theorem 1(ii).

4. Conclusion and Suggestions

In this paper, in the section named "Automorphisms with Lie İdeals", algebraic identities, including derivations on the prime ring, discuss, and we obtain derivations information.

We also examine algebraic identities involving derivations on the Lie ideal of the prime ring. We prove that the Lie ideal, which satisfies the identities discussed in the section, is obtained in the center of the prime ring.

In future studies, the hypotheses in this study can examine using homoderivations or semiderivations of the prime ring.

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Statement of Research and Publication Ethics

The study is complied with research and publication ethics

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