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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 3, Pages 710–719 (2022)
DOI:10.31801/cfsuasmas.1030942
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: December 1, 2021; Accepted: March 12, 2022

A DIFFERENT APPROACH TO BOUNDEDNESS OF THE $B-$MAXIMAL OPERATORS ON THE VARIABLE LEBESGUE SPACES

Esra KAYA

Department of Mathematics, Faculty of Science, Bilecik Seyh Edebali University,
11230 Bilecik, TURKEY

Abstract. By using the $L_{p(\cdot)}-$boundedness of a maximal operator defined on homogeneous space, it has been shown that the $B-$maximal operator is bounded. In the present paper, we aim to bring a different approach to the boundedness of the $B-$maximal operator generated by generalized translation operator under a continuity assumption on $p(\cdot)$. It is noteworthy to mention that our assumption is weaker than uniform Hölder continuity.

1. Introduction

Nowadays, there is a big attention on the singular integral operator and maximal operators which are defined on variable Lebesgue spaces. The problem that such operators are bounded under which conditions is well-studied and it is the main topic of harmonic analysis. $L_{p(\cdot)}-$boundedness of the Hardy-Littlewood maximal operator and singular integral operators have been investigated in [1–5].

This study is dealing with the boundedness of maximal operator generated by the Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad 1 \le k \le n,$$

which has big importance in harmonic analysis. In [8], Guliyev has obtained the $L_{p,\gamma}-$boundedness of the $B-$maximal operator. Moreover, in [6, 12], it has been shown that the $B-$maximal operator is $L_{p(\cdot),\gamma}-$bounded by using the $L_{p(\cdot)}-$ boundedness of a maximal operator whose domain is a homogeneous space.

2020 *Mathematics Subject Classification.* Primary 42B25; Secondary 42A50, 42B35.
Keywords. Maximal operator, singular integrals, variable Lebesgue space.
esra.kaya@bilecik.edu.tr; 0000-0002-6971-0452.

In this study, we obtain that the $B-$maximal operator is bounded on the variable Lebesgue spaces. Here, there are some difficulties while studying the theory of variable Lebesgue spaces. One of them, the generalized translation operator is in general not continuous on the spaces $L_{p(\cdot),\gamma}$. Particularly, if $p(\cdot)$ is not constant, then the generalized translation operator T^y is not continuous on the variable Lebesgue spaces. But, it is still possible to overcome these difficulties by taking some regularity conditions on this exponent function. In [7], it has been obtained that the generalized translation operator on the spaces $L_{p(\cdot),\gamma}$ is bounded. The construction of the article is as follows: The first section is devoted to introduction. In the second section, we recall some basic concepts, notations and some known results which we need throughout the paper. In the third section, we present that the $B-$maximal operator on the spaces $L_{p(\cdot),\gamma}$ is bounded under suitable assumptions by a different approach.

2. Preliminaries

Now, we pause to collect some basic concepts, notations and known results which are beneficial for us.

Let $x = (x', x'')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_k > 0, 1 \le k \le n\}$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, and $S_+ = \{x \in \mathbb{R}^n_{k,+} : |x| = 1\}$. Denote by $B_+(x, r)$ the open ball of radius r centered at x, namely,
$B_+(x, r) = \{y \in \mathbb{R}^n_{k,+} : |x - y| < r\}$. Let $B_+(0, r) \subset \mathbb{R}^n_{k,+}$ be a measurable set, then

$$|B_+(0,r)|_\gamma = \int_{B_+(0,r)} (x')^\gamma dx = \omega(n,k,\gamma) r^{n+|\gamma|},$$

where $\omega(n,k,\gamma) = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left(\frac{\gamma_i}{2}\right)}$.

We will now introduce the spaces $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and recall the basic properties of it. Let $\mathcal{P}(\mathbb{R}^n_{k,+})$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty]$. The elements of $\mathcal{P}(\mathbb{R}^n_{k,+})$ are called variable exponent functions and also let

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n_{k,+}} p(x), \qquad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} p(x).$$

Given $p(\cdot)$, the conjugate exponent function is as follows:

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}^n_{k,+}.$$

The analog of log-Hölder continuity for variable Lebesgue spaces related to the Laplace-Bessel differential operator is defined by the following.

Definition 1. *Given a function $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$, $p(\cdot)$ is called log-Hölder continuous on $\mathbb{R}^n_{k,+}$, if there exist constants $C_0, C_\infty > 0$ and p_∞ such that for all $|x-y| \leq \dfrac{1}{2}$, and $x, y \in \mathbb{R}^n_{k,+}$,*

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x-y|}, \tag{1}$$

and

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \tag{2}$$

where $p_\infty = \lim\limits_{x\to\infty} p(x) > 1$. If (1) and (2) hold for $p(\cdot)$, then it is denoted by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$, and $p(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n_{k,+})$, respectively.

Lemma 1. *[7] Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$ be continuous. The followings are equivalent:*

(i) *$p(\cdot)$ is uniformly continuous with $|p(x) - p(y)| \leq \dfrac{C_0}{\ln|x-y|^{-1}}$ for all $0 < |x-y| \leq \frac{1}{2}$.*

(ii) *$|B_+|_\gamma^{p_- - p_+} \leq C_1$ holds for all open balls B_+.*

The space $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is known as the set of measurable functions f such that for a variable exponent $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty]$,

$$\|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),\gamma}(f/\lambda) \leq 1\right\} < \infty,$$

where

$$\rho_{p(\cdot),\gamma} := \int_{\mathbb{R}^n_{k,+}} |f(x)|^{p(x)} (x')^\gamma dx.$$

Note that the variable Lebesgue space $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is a Banach space for $1 < p_- \leq p(x) \leq p_+ < \infty$.

The definition of the generalized translation operator is as follows:

$$T^y f(x) := C_{\gamma,k} \int_0^\pi \cdots \int_0^\pi f\left[(x_1, y_1)_{\alpha_1}, \ldots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] d\gamma(\alpha),$$

where $C_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma(\frac{\gamma_i+1}{2})[\Gamma(\frac{\gamma_i}{2})]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \leq i \leq k$, $1 \leq k \leq n$, and $d\gamma(\alpha) = \prod\limits_{i=1}^k \sin^{\gamma_i - 1}\alpha_i \, d\alpha_i$ [13, 14]. Notice that the generalized translation operator is related to the Laplace-Bessel differential operator.

The definition of the $B-$convolution operator is as follows:

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\gamma dy.$$

Given a function $f \in L^{\mathrm{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, then the maximal operator associated with the Laplace-Bessel differential operator ($B-$maximal operator) (see [8]) is as follows:

$$M_\gamma f(x) = \sup_{r>0} |B_+(0,r)|^{-1}_\gamma \int_{B_+(0,r)} T^y|f(x)|(y')^\gamma dy.$$

Let $B_+ \in \mathbb{R}^n_{k,+}$ be an arbitrary ball and $f \in L^{\mathrm{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, then define

$$M_{\gamma,B_+} f := |B_+(0,r)|^{-1}_\gamma \int_{B_+} T^y|f(x)|(y')^\gamma dy.$$

By taking supremum over all balls centered at x, one can easily observe that

$$M_\gamma f := \sup_{B_+(x)} M_{\gamma,B_+(x)} f.$$

As mentioned earlier, the variable Lebesgue spaces $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ have some undesired properties about the generalized translation operator. In order to overcome this problem, it is necessary to give some smoothness conditions on $p(\cdot)$. The following theorem states the necessary condition for the boundedness of generalized translation operator.

Theorem 1. *[7] Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$ with $1 < p_- \le p_+ < \infty$. Then for all $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+}) \cap \mathcal{S}'_+(\mathbb{R}^n_{k,+})$ with* supp $F_B f \subset \{\xi \in \mathbb{R}^n_{k,+} : |\xi| \le 2^{v+1}\}$, $v \in \mathbb{N}_0$,

$$\|T^y f(x)\|_{p(\cdot),\gamma} \le c \exp\left((2 + 2^{vn}|y|)c_{\log}(p)\right) \|f\|_{p(\cdot),\gamma},$$

holds, where $c > 0$ is independent of v.

3. Main Results

This section is devoted to our main results. First of all we obtain some lemmas which we need to prove that the $B-$maximal operator is bounded on variable Lebesgue spaces.

Lemma 2. *Let $p(\cdot) \in \mathbb{R}^n_{k,+}$ be as in Lemma 1. Then there exists a positive constant $C(p,\gamma) > 0$ such that*

$$(M_\gamma f(x))^{\frac{p(x)}{p_-}} \le C(p,\gamma)\left(M_\gamma(|f|^{\frac{p(\cdot)}{p_-}})(x) + 1\right), \quad \textit{for all} \quad x \in \mathbb{R}^n_{k,+},$$

holds for all $\|f\|_{p(\cdot),\gamma} \le 1$.

Proof. Define $q(\cdot) := \dfrac{p(\cdot)}{p_-}$, then $q(\cdot)$ is also as in Lemma 1. Let $\|f\|_{p(\cdot),\gamma} \le 1$, then $\rho_{p(\cdot),\gamma}(f) \le 1$. By Theorem 1, for $r \ge \dfrac{1}{2}$, we get

$$(M_\gamma f)^{q(x)} = \left(|B_+|^{-1}_\gamma \int_{B_+} T^y|f(x)|(y')^\gamma dy\right)^{q(x)}$$

$$
\begin{aligned}
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}\left(\frac{1}{p(y)}T^y|f(x)|^{p(y)}(y')^\gamma+\frac{1}{p'(y)}(y')^\gamma\right)dy\right)^{q(x)}\\
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}\frac{1}{p(y)}T^y|f(x)|^{p(y)}(y')^\gamma dy+|B_+|_\gamma^{-1}\int_{B_+}\frac{1}{p'(y)}(y')^\gamma dy\right)^{q(x)}\\
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{p(y)}(y')^\gamma dy+|B_+|_\gamma^{-1}\int_{B_+}(y')^\gamma dy\right)^{q(x)}\\
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}|f(x)|^{p(y)}(y')^\gamma dy+|B_+|_\gamma^{-1}\int_{B_+}(y')^\gamma dy\right)^{q(x)}\\
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}(|f(x)|^{p(y)}+1)(y')^\gamma dy\right)^{q(x)}\\
&\le \left(|B_+|_\gamma^{-1}\rho_{p(\cdot),\gamma}(f)+1\right)^{q(x)}\\
&\le \left(|B_+(0,\frac{1}{2})|_\gamma^{-1}+1\right)^{q_+}.
\end{aligned}
$$

If $0<r<\dfrac{1}{2}$, then $|B_+|_\gamma\le(2r)^{n+|\gamma|}<1$, and

$$
\begin{aligned}
(M_\gamma f)^{q(x)} &= \left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|(y')^\gamma dy\right)^{q(x)}\\
&\le \left[\left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{q_-}(y')^\gamma dy\right)^{\frac{1}{q_-}}\left(|B_+|_\gamma^{-1}\int_{B_+}(y')^\gamma dy\right)^{\frac{1}{q'_-}}\right]^{q(x)}\\
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{q_-}(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}\\
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \left(|B_+|_\gamma^{-1}\int_{B_+}(T^y|f(x)|^{q(y)}+1)(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}\\
&\le |B_+|_\gamma^{-\frac{q(x)}{q_-}}3^{q_+}\left(\frac{1}{3}\int_{B_+}(T^y|f(x)|^{q(y)}+1)(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}.
\end{aligned}
$$

Since,

$$\begin{aligned}\frac{1}{3}\int_{B_+}(T^y|f(x)|^{q(y)}+1)(y')^\gamma dy &\le \frac{1}{3}\int_{B_+}(T^y|f(x)|^{p(y)}+2)(y')^\gamma dy\\ &\le \frac{1}{3}\int_{B_+}T^y|f(x)|^{p(y)}(y')^\gamma dy+\frac{2}{3}|B_+|_\gamma<1,\end{aligned}$$

and from Lemma 1, we obtain

$$\begin{aligned}(M_\gamma f)^{q(x)} &\le |B_+|_\gamma^{-\frac{q(x)}{q_-}}3^{q_+}\left(\frac{1}{3}\int_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy+\frac{2}{3}|B_+|_\gamma\right)\\ &\le |B_+|_\gamma^{-\frac{q(x)}{q_-}}|B_+|_\gamma 3^{q_+-1}\left(\int_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy+2\right)\\ &\le |B_+|_\gamma^{\frac{q_--q_+}{q_-}}3^{q_+-1}\left(\oint_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy+2\right)\\ &\le C_0\,3^{q_+-1}(M_\gamma(|f|^{q(y)})+2).\end{aligned}$$

If one takes supremum over all balls B_+, then the proof is completed. $\square$

Lemma 3. *Let $p(\cdot)\in\mathbb{R}^n_{k,+}$ be as in Lemma 1 and be constant outside some ball $B_+(0,r)$. Then there exist a constant $C(p,\gamma)>0$, and $h\in L_{1,\infty,\gamma}(\mathbb{R}^n_{k,+})\cap L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ such that*

$$(M_\gamma f(x))^{\frac{p(x)}{p_-}}\le C(p,\gamma)\,M_\gamma\left(|f|^{\frac{p(\cdot)}{p_-}}\right)(x)+h(x)\quad \textit{for a.a. } x\in\mathbb{R}^n_{k,+},$$

holds for all $\|f\|_{p(\cdot),\gamma}\le 1$.

Proof. Define $q(\cdot):=\dfrac{p(\cdot)}{p_-}$, and $q_\infty:=\dfrac{p_\infty}{p_-}$, then $q(\cdot)$ satisfies the equivalent conditions of Lemma 1. Let $\|f\|_{p(\cdot),\gamma}\le 1$, then $\rho_{p(\cdot),\gamma}(f)\le 1$. Split $f=f_0+f_1$ such that $f_0:=\chi_{B_+}f$, and $f_1:=\chi_{\mathbb{R}^n_{k,+}\backslash B_+}f$. Thus, for all $x\in B_+(0,2r)$,

$$(M_\gamma f(x))^{q(x)}\le C(q,\gamma)\,(M_\gamma(|f|^{q(\cdot)})+1). \tag{3}$$

Now let $x\in\mathbb{R}^n_{k,+}\backslash B_+(0,2r)$. Then $|x|-r\ge\dfrac{1}{2}|x|$, and $|B_+(x,|x|-r)|_\gamma\ge C\,|x|^{n+|\gamma|}$. Since $\operatorname{supp} f_0\subset B_+(x,r)$, and from Theorem 1, we get

$$\begin{aligned}(M_\gamma f_0(x))^{q(x)} &\le \left(\sup_{|x|-r<r}|B_+(x,r)|_\gamma^{-1}\int_{B_+(x,r)}T^y|f_0(x)|(y')^\gamma dy\right)^{q(x)}\\ &\le \left(|B_+(x,|x|-r)|_\gamma^{-1}\int_{B_+(x,r)}T^y|f(x)|(y')^\gamma dy\right)^{q(x)}\end{aligned}$$

$$\leq \left(C\,|x|^{-n-|\gamma|}\int_{B_+(x,r)} T^y|f(x)|(y')^\gamma dy\right)^{q(x)}$$
$$\leq \left(C\,|x|^{-n-|\gamma|}\int_{B_+(x,r)} |f(x)|(y')^\gamma dy\right)^{q(x)}$$
$$\leq \left(C\,|x|^{-n-|\gamma|}\int_{B_+(x,r)} (|f(x)|^{p(y)}+1)(y')^\gamma dy\right)^{q(x)}$$
$$\leq \left(C\,|x|^{-n-|\gamma|}\rho_{p(\cdot),\gamma}(f)\right)^{q(x)}$$
$$\leq C(q,\gamma)|x|^{-n-|\gamma|}. \tag{4}$$

Moreover, for $x \in \mathbb{R}^n_{k,+}\backslash B_+(0,2r)$,

$$\begin{aligned}(M_\gamma f_1(x))^{q(x)} &= \left(\fint_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)} |T^y f_1(x)|(y')^\gamma dy\right)^{q(x)}\\
&\leq \left(\fint_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)} |T^y f_1(x)|(y')^\gamma dy\right)^{q_\infty}\\
&\leq \fint_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)} T^y|f_1(x)|^{q_\infty}(y')^\gamma dy\\
&\leq \fint_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)} T^y|f_1(x)|^{q(x)}(y')^\gamma dy\\
&\leq M_\gamma(|f|^{q(x)})(x). \end{aligned}\tag{5}$$

By (3), (4) and (5), we obtain

$$\begin{aligned}(M_\gamma f(x))^{q(x)} &\leq \chi_{B_+(0,2r)}\,(M_\gamma f(x))^{q(x)} + \chi_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)}\,(M_\gamma f_0(x)+M_\gamma f_1(x))^{q(x)}\\
&\leq \chi_{B_+(0,2r)}\,(M_\gamma f(x))^{q(x)} + C(q,\gamma)\chi_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)}\left((M_\gamma f_0(x))^{q(x)}+(M_\gamma f_1(x))^{q(x)}\right)\\
&\leq C(q,\gamma)\,M_\gamma(|f|^{q(\cdot)})(x) + \chi_{B_+(0,2r)}C(q,\gamma)\\
&+\left(\sup_{x\in\mathbb{R}^n_{k,+}\backslash B_+(0,2r)}\fint_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)}(y')^\gamma dy\right)^{q(x)}\\
&\leq C(q,\gamma)\,M_\gamma(|f|^{q(\cdot)})(x) + \underbrace{\chi_{B_+(0,2r)}C(q,\gamma)+\chi_{\mathbb{R}^n_{k,+}\backslash B_+(0,2r)}C(q,\gamma)\,|x|^{-n-|\gamma|}}_{=:h},\end{aligned}$$

for all $x \in \mathbb{R}^n_{k,+}$. The fact that $h \in L_{1,\infty,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ proves the lemma. $\square$

Now we can present our main theorem.

Theorem 2. *Let $p(\cdot)$ be as in Lemma 3 with $p_- > 1$. Then M_γ is bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, i.e.*

$$\|M_\gamma f\|_{p(\cdot),\gamma} \le C(p,\gamma)\,\|f\|_{p(\cdot),\gamma}.$$

Proof. Since $M_\gamma(\lambda f) = \|\lambda\| M_\gamma f$, we have $\|M_\gamma f\|_{p(\cdot),\gamma} \le C$, for all $\|f\|_{p(\cdot),\gamma} \le 1$. Since $p_+ < \infty$, it is sufficient to illustrate $\rho_{p(\cdot),\gamma}(M_\gamma f) \le C$ for all $\|f\|_{p(\cdot),\gamma} \le 1$. Let $f \in L_{p(\cdot),\gamma}$ with $\|f\|_{p(\cdot),\gamma} \le 1$. Then $\rho_{p(\cdot),\gamma}(M_\gamma f) \le 1$. Moreover, let $q(\cdot) := p(\cdot)/p_-$. By Lemma 3, there exists $h \in L_{1,\infty,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ such that $(M_\gamma f)^{q(\cdot)} \le C(p,\gamma)\, M_\gamma(|f|^{q(\cdot)}) + h$. Thus,

$$\begin{aligned}
\rho_{p(\cdot),\gamma}(M_\gamma f) &= \int_{\mathbb{R}^n_{k,+}} |M_\gamma f|^{p(x)} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\sup_{B_+} \int_{B_+} T^y |f(x)| (y')^\gamma dy\right)^{p(x)} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\sup_{B_+} \int_{B_+} T^y |f(x)| (y')^\gamma dy\right)^{q(x)p_-} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\left(\sup_{B_+} \int_{B_+} T^y |f(x)| (y')^\gamma dy\right)^{q(x)}\right)^{p_-} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(|M_\gamma f|^{q(x)}\right)^{p_-} (x')^\gamma dx \\
&= \left\|(M_\gamma f)^{q(x)}\right\|_{p_-,\gamma}^{p_-} \\
&\le \left(C(p,\gamma)\, \left\|M_\gamma(|f|^{q(x)})\right\|_{p_-,\gamma} + \|h\|_{p_-,\gamma}\right)^{p_-}
\end{aligned}$$

holds and since $p_- > 1$, one can see that the $B-$maximal operator $M_\gamma f$ is continuous on $L_{p_-,\gamma}(\mathbb{R}^n_{k,+})$. Therefore, we obtain that

$$\begin{aligned}
\rho_{p(\cdot),\gamma}(M_\gamma f) &\le \left(C(p,\gamma)\, \|M_\gamma(|f|^{q(x)})\|_{p_-,\gamma} + \|h\|_{p_-,\gamma}\right)^{p_-} \\
&= \left(C(p,\gamma)\, \rho_{p(\cdot),\gamma}(f)^{\frac{1}{p_-}} + \|h\|_{p_-,\gamma}\right)^{p_-} \le C(p,\gamma),
\end{aligned}$$

and this completes the proof. □

4. Concluding Remarks

The Hardy-Littlewood maximal operators, singular integral operators, rough integral operator, its commutators and their boundedness on the various function spaces are crucial topics of Harmonic Analysis. In this study, we have shown that the $B-$maximal operator on the variable Lebesgue spaces is bounded under suitable assumptions by a different approach. The boundedness of this operator plays a significant role in order to obtain the boundedness of the singular integral operator, fractional integral operator and its commutators. The fractional versions of these operators have recently become an active area of research (see [9–11, 15, 16]). As a future direction of this study, one might extend to the case that the Laplace-Bessel differential operators with coefficient such as $a(x)$ that could be continuous or Vanishing Mean Oscillation functions.

Declaration of Competing Interests The author declares no competing interests.

Acknowledgement The authors thank the referees for their valuable suggestions and comments.

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