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Research Article

On the Geometry (k,μ)-Paracontact Metric Manifold Satisfying Certain Curvature Conditions

Pakize UYGUN [1] , Mehmet ATÇEKEN [2]

Abstract
In the present paper, we have studied the curvature tensors of (k,μ)-paracontact metric manifold satisfying the conditions $\widetilde{Z}(X,Y)\cdot R=0$, $\widetilde{Z}(X,Y)\cdot\widetilde{Z}=0$, $R(X,Y)\cdot\widetilde{Z}=0$ and $R(X,Y)\cdot R=0$. According the cases, we have classified (k,μ)-paracontact metric manifolds.

Keywords and 2010 Mathematics Subject Classification
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[1] *Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, 60100, Tokat, Turkey,*
[2] *Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, 68100, Aksaray, Turkey*
[1] ✉pakizeuygun@hotmail.com, [2] ✉mehmet.atceken382@gmail.com,
Corresponding author: Pakize Uygun
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1. Introduction

The notion of paracontact geometry was introduced by Kaneyuki and Williams in 1985 [6]. A systematic study of paracontact metric manifolds and their subclasses was carried out by Zamkovoy [15]. Several geometers studied paracontact metric manifolds and obtain various important properties of these manifolds.

[5] introduced a new type of paracontact geometry, so called paracontact metric (k,μ)-spaces, where k and μ are real constants. Such manifolds are known as (k,μ)-paracontact metric manifolds. The class of (k,μ)-paracontact metric manifolds contains para-Sasakian manifolds.

As a generalization of locally symmetric spaces, many authors have studied semi-symmetric spaces. A semi-Riemannian manifold (M^{2n+1},g), $n\geq 1$, is said to be semi-symmetric if its curvature tensor R satisfies $R(X,Y)\cdot R=0$, where $R(X,Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors X,Y [9, 7]. A manifold is said to be Ricci semisymmetric if $R(X,Y)\cdot S=0$ where S denotes the Ricci tensor of type (0,2). A general classification of these manifolds has been worked out by Mirzoyan [8].

The concept of locally ϕ-symmetric was introduced by Takahashi [10] in Sasakian geometry as a weaker version of locally symmetric manifolds. Atçeken M. [2]. studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor. Arslan et. al. produced the works on contact manifold curvature tensor [1].

Motivated by the studies of the above authors, in this paper we classify (k,μ)-paracontact manifolds, which satisfy the curvature conditions $\widetilde{Z}(X,Y)\cdot\widetilde{Z}=0$, $\widetilde{Z}(X,Y)\cdot R=0$, $R(X,Y)\cdot\widetilde{Z}=0$ and $R(X,Y)\cdot R=0$ where $\widetilde{Z}$ is the concircular curvature tensor, R is the Riemannian curvature tensor.

2. Preliminaries

A contact manifold is a $C^{\infty}-(2n+1)$manifold M^{2n+1}equipped with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on M^{2n+1}. Given such a form η, it is well known that there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field X on M^{2n+1}. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type $(1,1)$ such that

$$\phi^2X = X-\eta(X)\xi, \quad \eta(\xi)=1, \quad \eta\circ\phi=0, \quad \phi\xi=0, \tag{1}$$

$$g(\phi X,\phi Y) = -g(X,Y)+\eta(X)\eta(Y), \quad g(X,\xi)=\eta(X) \tag{2}$$

for all vector fields X,Y on M. Then the structure (ϕ,ξ,η,g) on M is called a paracontact metric structure and the manifold equipped with such a structure is called a almost paracontact metric manifold [15].

We now define a $(1,1)$ tensor field h by $h=\frac{1}{2}L_\xi\phi$, where L denotes the Lie derivative, then h is symmetric and satisfies [15].

$$h\phi=-\phi h, \quad h\xi=0, \quad Tr.h=Tr.\phi h=0. \tag{3}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\nabla_X\xi = -\phi X+\phi hX \tag{4}$$

for all $X\in\chi(M)$ [15]. For a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, if ξ is a killing vector field or equivalently, $h=0$ it is called a K-paracontact manifold.

A paracontact metric structure (ϕ,ξ,η,g) is normal, that is, satisfies $[\phi,\phi]+2d\eta\otimes\xi=0$, which is equivalent to

$$(\nabla_X\phi)Y=-g(X,Y)\xi+\eta(Y)X$$

for any $X,Y\in\chi(M)$ [15]. If an almost paracontact metric manifold is normal, than it called paracontact metric manifold. Any para-Sasakian manifold is K-paracontact, and the converse holds when $n=1$, that is, for 3-dimensional spaces. Any para-Sasakian manifold satisfies

$$R(X,Y)\xi = -(\eta(Y)X-\eta(X)Y) \tag{5}$$

for any $X,Y\in\chi(M)$, but this is not a sufficient condition for a paracontact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact, but the converse is not always true [4].

A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ is of the from $S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y)$, where a,b are smooth functions on M. If $b=0$, then the manifold is also called Einstein [11].

A paracontact metric manifold is said to be a (k,μ)- paracontact manifold if the curvature tensor R satisfies

$$\widetilde{R}(X,Y)\xi = k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY] \tag{6}$$

for all $X,Y\in\chi(M)$ and k,μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X,Y)\xi=0$ [16].

In particular, if $\mu=0$, then the paracontact metric (k,μ)-manifold is called paracontact metric $N(k)$-manifold . Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k(\eta(Y)X-\eta(X)Y) \tag{7}$$

for all $X,Y\in\chi(M)$. Though the geometric behavior of paracontact metric (k,μ)-spaces is different according as $k<-1$, or $k>-1$, but there are also some common results for $k<-1$ and $k>-1$.

Lemma 1. *There does not exist any paracontact (k,μ)-manifold of dimension greater than 3 with $k>-1$ which is Einstein whereas there exits such manifolds for $k<-1$ [5].*

In a paracontact metric (k,μ)-manifold $(M^{2n+1}\phi,\xi,\eta,g)$, $n>1$, the following relations hold:

$$h^2=(k+1)\phi^2, \tag{8}$$

$$(\widetilde{\nabla}_X \phi)Y = -g(X-hX,Y)\xi + \eta(Y)(X-hX), \text{ for } k \neq -1, \tag{9}$$

$$S(X,Y) \quad = \quad [2(1-n)+n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y)+[2(n-1)+n(2k-\mu)]\eta(X)\eta(Y), \tag{10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{11}$$

$$QY \quad = \quad [2(1-n)+n\mu]Y+[2(n-1)+\mu]hY+[2(n-1)+n(2k-\mu)]\eta(Y)\xi, \tag{12}$$

$$Q\xi = 2nk\xi, \tag{13}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi \tag{14}$$

for any vector fields X,Y on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1},g), respectively [5].

Let (M,g) be an $(2n+1)$-dimensional Riemanian manifold. Then the concircular curvature tensor $\widetilde{Z}$ is defined by

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{15}$$

for all $X,Y,Z \in \chi(M)$, where r is the scalar curvature of M and where R denotes the Riemannian curvature tensor of M and Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$ [11].

3. A (k,μ)- Paracontact Metric Manifold Satisfying Certain Curvature Tensor Conditions

In this section, we will give the main results for this paper.

Let M be $(2n+1)$-dimensional (k,μ)-paracontact metric manifold and we denote the Riemannian curvature tensor of R, then from (6), we have

$$R(\xi,Y)Z = k(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY). \tag{16}$$

In (6), choosing $Y = \xi$, we obtain

$$R(X,\xi)\xi = k(X - \eta(X)\xi) + \mu hX. \tag{17}$$

Also from (16), we get

$$\eta(R(\xi,Y)Z) = k(g(Y,Z) - \eta(Y)\eta(Z)) + \mu g(hY,Z) \tag{18}$$

from which (15), we have

$$\widetilde{Z}(\xi,Y)Z = (k - \frac{r}{2n(2n+1)})(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY) \tag{19}$$

and

$$\widetilde{Z}(\xi,Y)\xi = (k - \frac{r}{2n(2n+1)})(\eta(Y)\xi - Y) - \mu hY. \tag{20}$$

Theorem 2. *Let M be a $(2n+1)$-dimensional (k,μ)-paracontact manifold. Then $R(X,Y)\cdot\widetilde{Z}=0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $R(X,Y)\cdot\widetilde{Z}=0$. Then $X=\xi$, we have

$$\begin{aligned}(R(\xi,Y)\widetilde{Z})(U,W)Z &= R(\xi,Y)\widetilde{Z}(U,W)Z-\widetilde{Z}(R(\xi,Y)U,W)Z\\ &\quad -\widetilde{Z}(U,R(\xi,Y)W)Z-\widetilde{Z}(U,W)R(\xi,Y)Z=0,\end{aligned} \tag{21}$$

for any $Y,U,W,Z\in\chi(M)$. Using (16), (19), $Z=\xi$ in (21) and set $k-\frac{r}{2n(2n+1)}=A$, we obtain

$$\begin{aligned}(R(\xi,Y)\widetilde{Z})(U,W)\xi &= R(\xi,Y)(A(\eta(W)-\eta(U))+\mu(\eta(W)hU-\eta(U)hW)\\ &\quad -\widetilde{Z}(k(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi-\eta(U)hY),W)\xi\\ &\quad -\widetilde{Z}(U,k(g(Y,W)\xi-\eta(W)Y)+\mu(g(hY,W)\xi-\eta(W)hY)\xi\\ &\quad -\widetilde{Z}(U,W)(k(\eta(Y)\xi-Y)-\mu hY)=0.\end{aligned} \tag{22}$$

Here setting by using (17), (6) and inner product both sides of (22) by $V\in\chi(M)$, we obtain

$$\begin{aligned}&kg(\widetilde{Z}(U,W)Y,V)+\mu g(\widetilde{Z}(U,W)hY,V)\\ &+k\mu\eta(W)\eta(V)g(Y,hU)+\mu^2(k+1)\eta(W)\eta(V)g(Y,U)\\ &-k\mu\eta(U)\eta(V)g(Y,hW)-\mu^2(k+1)\eta(U)\eta(V)g(Y,W)\\ &+Akg(Y,U)g(W,V)+k\mu g(Y,U)g(hW,V)\\ &+A\mu g(hY,U)g(W,V)+\mu^2 g(hY,U)g(hW,V)\\ &-Akg(Y,W)g(U,V)-k\mu g(Y,W)g(hU,V)\\ &-A\mu g(hY,W)g(U,V)-\mu^2 g(hY,W)g(hU,V)=0.\end{aligned} \tag{23}$$

Using the equations (1), (8) and choosing $U=V=e_i$, ξ, in (23), $1\le i\le n$, for orthonormal basis of $\chi(M)$, we arrive

$$kS(W,Y)+\mu S(W,hY)-2nk^2 g(W,Y)-2nk\mu g(W,hY)=0. \tag{24}$$

In (24), using (15) and replacing hY of Y, we get

$$kS(W,hY)+\mu(1+k)S(W,Y)-2nk^2 g(W,hY)-2nk\mu(1+k)g(W,Y)=0. \tag{25}$$

Also from (24) and (25), we conclude

$$S(Y,W)=2nkg(Y,W).$$

So, M is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold i.e. $S(Y,W)=2nkg(Y,W)$, then from (25), (24), (23), (22) and (21), we have $R(X,Y)\cdot\widetilde{Z}=0$. ∎

Theorem 3. *Let M be a $(2n+1)$-dimensional (k,μ)-paracontact manifold. Then, $\widetilde{Z}(X,Y)\cdot\widetilde{Z}=0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $\widetilde{Z}(X,Y)\cdot\widetilde{Z}=0$. Then we have,

$$\begin{aligned}(\widetilde{Z}(X,Y)\widetilde{Z})(U,W)Z &= \widetilde{Z}(X,Y)\widetilde{Z}(U,W)Z-\widetilde{Z}(\widetilde{Z}(X,Y)U,W)Z\\ &\quad -\widetilde{Z}(U,\widetilde{Z}(X,Y)W)Z-\widetilde{Z}(U,W)\widetilde{Z}(X,Y)Z=0\end{aligned} \tag{26}$$

for any $X,Y,U,W,Z\in\chi(M)$. Using (19), (20) and $X=Z=\xi$ in (26), for $k-\frac{r}{2n(2n+1)}=A$, we obtain

$$\begin{aligned}(\widetilde{Z}(\xi,Y)\widetilde{Z})(U,W)\xi &= \widetilde{Z}(\xi,Y)(A(\eta(W)-\eta(U))+\mu(\eta(W)hU-\eta(U)hW)\\ &\quad -\widetilde{Z}(A(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi-\eta(U)hY),W)\xi\\ &\quad -\widetilde{Z}(U,A(g(Y,W)\xi-\eta(W)Y)+\mu(g(hY,W)\xi-\eta(W)hY)\xi\\ &\quad -\widetilde{Z}(U,W)(A(\eta(Y)\xi-Y)-\mu hY)=0.\end{aligned} \tag{27}$$

Here setting by using (19) and inner product both sides of (27) by $V \in \chi(M)$, we get

$$\begin{aligned}
&Ag(\widetilde{Z}(U,W)Y,V)+\mu g(\widetilde{Z}(U,W)hY,V)+A\mu(\eta(W)\eta(V)g(Y,hU)\\
&-\eta(U)\eta(V)g(Y,hW))+\mu^2(k+1)(\eta(W)\eta(V)g(Y,U)\\
&-\eta(U)\eta(V)g(Y,W))+Akg(Y,U)g(W,V)+A\mu(g(Y,U)g(hW,V)\\
&+g(hY,U)g(W,V))+\mu^2(g(hY,U)g(hW,V)-g(hY,W)g(hU,V))\\
&-A^2g(Y,W)g(U,V)-A\mu(g(Y,W)g(hU,V)+g(hY,W)g(U,V))=0.
\end{aligned} \tag{28}$$

Using the equations (1), (8), (15) and taking $U=V=e_i$, ξ in (28), $1\le i\le n$, for orthonormal basis of $\chi(M)$, we have

$$AS(W,Y)+\mu S(W,hY)-2nkAg(W,Y)-2nk\mu g(W,hY)=0. \tag{29}$$

In (29), replacing hY of Y, we arrive

$$AS(W,hY)+\mu(1+k)S(W,Y)-2nkAg(W,hY)-2nk\mu(1+k)g(W,Y)=0. \tag{30}$$

From (29) and (30), we conclude

$$S(W,Y)=2nkg(W,Y).$$

Thus, M is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold i.e. $S(Y,W)=2nkg(Y,W)$, then from (30), (29), (28), (27) and (26), we have $\widetilde{Z}(X,Y)\cdot\widetilde{Z}=0$. ∎

Theorem 4. *Let M be a $(2n+1)$-dimensional (k,μ)-paracontact manifold. Then, M is semisymetric if and only if M is an η-Einstein manifold.*

Proof. Suppose that M is semisymetric. Then we have,

$$(R(X,Y)R)(U,W)Z=R(X,Y)R(U,W)Z-R(R(X,Y)U,W)Z-R(U,R(X,Y)W)Z-R(U,W)R(X,Y)Z=0, \tag{31}$$

for any $X,Y,U,W,Z\in\chi(M)$. Using (16) and $X=Z=\xi$ in (31), we obtain

$$\begin{aligned}
(R(\xi,Y)R)(U,W)\xi \quad = \quad & R(\xi,Y)(k(\eta(W)-\eta(U))+\mu(\eta(W)hU-\eta(U)hW)\\
&-R(k(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi-\eta(U)hY),W)\xi\\
&-R(U,k(g(Y,W)\xi-\eta(W)Y)+\mu(g(hY,W)\xi-\eta(W)hY)\xi\\
&-R(U,W)(k(\eta(Y)\xi-Y)-\mu hY)=0.
\end{aligned} \tag{32}$$

By using (16), (6) and inner product both sides of (32) by $V\in\chi(M)$, we get

$$\begin{aligned}
&kg(R(U,W)Y,V)+\mu g(R(U,W)hY,V)+k\mu(\eta(W)\eta(V)g(Y,hU)\\
&-g(hY,W)g(U,V))+\mu^2(1+k)(\eta(U)\eta(V)g(Y,hU)\\
&-\eta(U)\eta(V)g(Y,W))+k^2(g(U,Y)g(V,W)-g(U,V)g(V,W))\\
&+k\mu(g(Y,U)g(hW,V)-\eta(U)\eta(V)g(Y,hW))+k\mu(g(hY,U)g(W,V)\\
&-g(Y,W)g(hU,V))+\mu^2(g(hY,U)g(hW,V)-g(hY,W)g(hU,V))=0.
\end{aligned} \tag{33}$$

Making use of (1), (11) and choosing $U=V=e_i$, ξ in (33), $1\le i\le n$, for orthonormal basis of $\chi(M)$, we have

$$kS(W,Y)+\mu S(W,hY)-2nk^2g(W,Y)-2nk\mu g(hY,W)+k\mu\eta(W)\eta(Y)=0. \tag{34}$$

In (34) using (8) and replacing hY of Y, we get

$$kS(W,hY)+\mu(1+k)S(W,Y)-2nk^2g(W,hY)-2nk\mu(1+k)g(U,V)=0. \tag{35}$$

From (34) and (35), we obtain

$$S(W,Y)=2nkg(W,Y)-\frac{k^2\mu}{(k^2-\mu^2)(1+k)}\eta(W)\eta(Y).$$

Thus, M is an η-Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an η-Einstein manifold i.e. $S(W,Y)=2nkg(W,Y)-\frac{k^2\mu}{(k^2-\mu^2)(1+k)}\eta(W)\eta(Y)$, then from (35), (34), (33), (32) and (31), we have $R(X,Y)\cdot R=0$. ∎

Theorem 5. *Let M be $(2n+1)$-dimensional a (k,μ)-paracontact manifold. Then, $\widetilde{Z}(X,Y)\cdot R=0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $\widetilde{Z}(X,Y)\cdot R=0$. Then we have,

$$\begin{aligned}(\widetilde{Z}(X,Y)R)(U,W)Z &= R(X,Y)R(U,W)Z-R(\widetilde{Z}(X,Y)U,W)Z\\ &\quad -R(U,\widetilde{Z}(X,Y)W)Z-R(U,W)\widetilde{Z}(X,Y)Z=0,\end{aligned} \tag{36}$$

for any $X,Y,U,W,Z\in\chi(M)$. Using (16), (20) and $X=Z=\xi$ in (36) for $k-\frac{r}{2n(2n+1)}=A$, we obtain

$$\begin{aligned}(\widetilde{Z}(\xi,Y)R)(U,W)\xi &= \widetilde{Z}(\xi,Y)(k(\eta(W)-\eta(U))+\mu(\eta(W)hU-\eta(U)hW)\\ &\quad -R(A(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi-\eta(U)hY),W)\xi\\ &\quad -R(U,A(g(Y,W)\xi-\eta(W)Y)+\mu(g(hY,W)\xi-\eta(W)hY)\xi\\ &\quad -R(U,W)(A(\eta(Y)\xi-Y)-\mu hY)=0.\end{aligned} \tag{37}$$

By using (19), (8) and inner product both sides of (37) by $V\in\chi(M)$, we obtain

$$\begin{aligned}&Ag(R(U,W)Y,V)+\mu g(R(U,W)hY,V)+A\mu\eta(W)\eta(V)g(Y,hU)\\ &+\mu^2(k+1)\eta(W)\eta(V)g(Y,U)-A\mu\eta(U)\eta(V)g(Y,hW)\\ &-\mu^2\eta(U)\eta(V)g(Y,W)+Akg(Y,U)g(W,V)+A\mu g(Y,U)g(hW,V)\\ &+k\mu g(hY,U)g(W,V)+\mu^2 g(hY,U)g(hW,V)-Akg(Y,W)g(U,V)\\ &-A\mu g(Y,W)g(hU,V)-k\mu g(hY,W)g(U,V)-\mu^2 g(hY,W)g(hU,V)=0.\end{aligned} \tag{38}$$

Using the equations (1), (11) and choosing $U=V=e_i$, ξ in (38), $1\le i\le n$, for orthonormal basis of $\chi(M)$, we have

$$AS(W,Y)+\mu S(W,hY)-2nAkg(W,Y)-2nk\mu g(W,hY)=0. \tag{39}$$

In (39), using the equations (8) and replacing hY of Y, we get

$$AS(W,hY)+\mu(1+k)S(W,Y)-2nkAg(W,hY)-2nk\mu(1+k)g(W,Y)=0. \tag{40}$$

Also from (39) and (40), we conclude

$$S(W,Y)=2nkg(W,Y).$$

Thus, M is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold i.e. $S(W,Y)=2nkg(W,Y)$, then from (40), (39), (38), (37) and (36), we have $\widetilde{Z}(X,Y)\cdot R=0$. ■

4. Conclusion

The methods we apply in this article can be applied to other manifolds as well. In general the results can be interpreted geometrically.

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