Turk. J. Math. Comput. Sci.
13(1)(2021) 145–161
© MatDer
DOI : 10.47000/tjmcs.757462

Numerical Analysis of Backward-Euler Method for Simplified MagnetoHydroDynamics (SMHD) with Linear Time Relaxation

Gamze Yüksel*, Mustafa Hİcret Yaman

Department of Mathematics, Faculty of Science, Mugla Sitki Kocman University, 48000, Mugla, Turkey.

Received: 24-06-2020 • Accepted: 15-04-2021

Abstract. In this study, the solutions of Simplified Magnetohyrodynamics (SMHD) equations by finite element method are examined with linear time relaxation term. Therefore, the differential filter $\kappa(u - \bar{u})$ term is added to SMHD equations for numerical regularization and it is introduced SMHD Linear Time Relaxation Model (SMHDL-TRM). The SMHDLTRM model is discretized by Backward-Euler (BE) method to obtain finite element solutions. The stability and convergency of the method is also conducted for SMHDLTRM. The present method is unconditionally stable and convergent with small time step condition. Additionally, the effectiveness of the method has presented with some numerical examples. The BE solutions of the SMHDLTRM are compared with the BE and the Crank-Nicolson (CN) solutions of the SMHD equations. This method can be applied to some problems when necessary more time steps to get accuracy or numerical solutions blow up for classical methods (BE or CN). All computations are conducted by using FreeFem++.

2010 AMS Classification: 65M12, 65N30, 76M10, 76W05

Keywords: MagnetoHydroDynamics, backward-Euler method, linear time relaxation, finite element method.

1. Introduction

Magnetohydrodynamics (MHD) is the physical-mathematical principal that concerns the dynamics of magnetic fields in electrically conducting fluids, e.g. in plasmas and liquid metals. The reduced form of MHD is obtained vanishing magnetic Reynolds number, which allows the fluid and magnetic equations to uncouple in MHD. The first study is given by Janet Peterson for the existence and uniqueness of the weak form of reduced MHD and a finite element algorithm is given for the approximate solution of it [18]. The error estimates are also derived in this study. In [8] Gunzburger, Meir and Peterson examine the equations of stationary, incompressible magneto-hydrodynamics posed in a bounded domain in three dimensions and they treat the full, coupled system of equations with inhomogeneous boundary conditions. There are several studies examined MHD in different perspectives. For instance, in [7] they examined deferred correction method for MHD system based on Elsässer variables. Layton et al. [15] and [16] introduced two partitioned methods to solve evolutionary MHD equations and provided a complete error analysis. Time discretization schemes for MHD problems were studied by Yuksel and Ingram in [23]. In another study, Belenli et al. [3] presented a numerical scheme for approximating solutions of incompressible MHD equations. Also, high order algebraic splitting methods for MHD flows are analyzed in [2]. Time discretization schemes for MHD problems were studied by Yuksel and Ingram in [22]. We consider simplified MHD (SMHD) equation in reduced form of MHD and we present a linear time relaxation model for SMHD.

*Corresponding Author
Email addresses: ngamze@mu.edu.tr (G. Yüksel), hicretyaman@gmail.com (M.H. Yaman)

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be an open, regular domain. The dimensionless quasi-static MHD is modelled by the system, see, e.g., [18]: Given time $T > 0$, body force f, interaction parameter $N > 0$, Hartmann number $M > 0$, and letting $\Omega_T := [0, T] \times \Omega$, find velocity $u : \Omega_T \to \mathbb{R}^d$, pressure $p : \Omega_T \to \mathbb{R}$, electric current density $j : \Omega_T \to \mathbb{R}^d$, magnetic field $B : \Omega_T \to \mathbb{R}^d$, and electric potential $\phi : \Omega_T \to \mathbb{R}$ satisfying

$$\begin{array}{rclrcl} N^{-1}(u_t + u \cdot \nabla u) & = & f + M^{-2}\Delta u - \nabla p + j \times B, & \nabla \cdot u & = & 0, \\ -\nabla\phi + u \times B & = & j, & \nabla \cdot j & = & 0, \\ \nabla \times B & = & R_m j, & \nabla \cdot B & = & 0. \end{array} \tag{1.1}$$

subject to boundary and initial conditions

$$\begin{array}{ll} u(x,t) = 0, & \forall (x,t) \in \partial\Omega \times [0,T], \\ \phi(x,t) = 0, & \forall (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), & \forall x \in \Omega. \end{array} \tag{1.2}$$

Here, $R_m = UL/\eta > 0$, U is the characteristic speed, L is the length of the problem domain, $\eta > 0$ is the magnetic diffusivity, $u_0 \in H_0^1(\Omega)^d$ and $\nabla \cdot u_0 = 0$. j and $\nabla \times B$ in (1.1)(3a) decouple $R_m << 1$. Supposing B is an applied (and known) magnetic field, (1.1) reduces to the simplified MHD (SMHD) system see, e.g., [18]: Find u, p, ϕ satisfying

$$\begin{array}{rcl} N^{-1}(u_t + u \cdot \nabla u) - M^{-2}\Delta u + \nabla p - j \times B & = & f, \\ \nabla \cdot u & = & 0, \\ -\Delta\phi + \nabla \cdot (u \times B) & = & 0 \end{array} \tag{1.3}$$

subject to (1.2). There are three distinct cases for $R_m << 1$.

i) The imposed magnetic field is static, the flow is induced by some external agency.

ii) The imposed magnetic field travels or rotates uniformly and slowly.

iii) The imposed magnetic field oscillates extremely rapidly.

Categories (i)-(iii) cover the majority of flows in engineering applications. For case (i), magnetic damping of jets, vortices or turbulence are typical examples. For case (ii), magnetic stirring using a rotating magnetic field is given and for case (iii), magnetic levitation can be given as an examples [6].

We consider a time relaxation model which is one of the regularization techniques. The time relaxation operator is introduced as a numerical regularization in [21]- [1] which are based on the study of Chapman–Enskog expansions by Rosenau [19], Schochet and Tadmor [20]. The time relaxation operator truncates small solution scales by injecting an extra dissipation to a simulation, without altering appreciably the solution's large scales [17]. Time relaxation is also known as nudging, damping or Newtonian damping in literature. The Time Relaxation Models (TRM) are introduced as regularization techniques for Navier–Stokes Equations (NSE) by Layton and Neda in [11]. The model is obtained by adding the relaxation term to momentum equations of the NSE. They also investigated selection of parameters of NSE for linear and nonlinear time relaxation models in that paper. Layton and Rebholz give the main theory about the time relaxation truncation scales in [14]. Also, Layton, Pruett and Rebholz [13] regularized temporally the flow by adding the filter term to the NSE. In [4], several published works are reviewed for development and implementations of time relaxation and time relaxation models (TRMs), and how such techniques are used to improve the accuracy and stability of fluid flow problems with higher Reynolds numbers are given by Breckling, Neda and Hill. In [17], Pakzad examined time averaged energy dissipation rate for the Time Relaxation Model for 3d turbulence with periodic boundary conditions. Also in [10], Işık et al. consider a second order time stepping finite element method for the Navier-Stokes equations. We also consider BE method for solutions of SMHD equation. BE discretization is examined in several studies. Işık [9] investigated spin up problem and accelerating convergence to steady state for Navier-Stokes (NSE) time relaxation model by using the BE time discretization and Özbunar [16] obtains stability and convergency analysis of the NSE time relaxation model. In one of the recent studies [5] it is presented that time filtered Backward Euler (BE) method delivers better correct energy and cross-helicity balance in comparison with the BE method.

In this study, we present the following model which is obtained by adding time relaxation term $\kappa(u - \bar{u})$ into the SMHD and it is introduced SMHD Linear Time Relaxation Model (SMHDLTRM);

$$\begin{array}{rcl} N^{-1}(u_t + u \cdot \nabla u) - M^{-2}\Delta u + \nabla p - j \times B + \kappa(u - \bar{u}) & = & f, \\ \nabla \cdot u & = & 0, \\ -\Delta\phi + \nabla \cdot (u \times B) & = & 0 \end{array} \tag{1.4}$$

where $\kappa, \delta > 0$ and $\bar{u}$ is the unique solution of

$$\bar{u}_t = \frac{(u-\bar{u})}{\delta} \ , \quad t>0,$$
$$\bar{u}(x,0) = u(x,0).$$

We consider a nonlinearly implicit Backward-Euler (BE) method for the solutions of SMHD equations. Let $0 = t_0 < t_1 < \ldots < t_K = T < \infty$ be a discretization of the time interval $[0,T]$ for a constant time step $\Delta t = t_n - t_{n-1}$. Write $z_n = z(t_n)$.

$$\begin{aligned} &N^{-1}\left(\tfrac{u^h_{n+1}-u^h_n}{\Delta t} + u_{n+1}.\nabla u^h_{n+1}\right) + M^{-2}\Delta u^h_{n+1} \\ &+\nabla p^h_{n+1} - \nabla\phi^h_{n+1} + u^h_{n+1}\times B = f_{n+1}, \\ &\nabla.u^h_{n+1} = 0, \\ &\nabla\phi^h_{n+1} - (u^h_{n+1}\times B) = 0, \\ &u^h(x,0) = u^h_0(x). \end{aligned}$$

Theorem 1.1 (Algorithm: Backward-Euler method for SMHD)**.** *Given $u_0 \in V$, find $(u^h_{n+1}, p^h_{n+1}, \phi^h_{n+1}) \in X^h \times Q^h \times S^h$ for each $n = 0,1,2,\ldots,K-1$ satisfying*

$$N^{-1}\left((\frac{u^h_{n+1}-u^h_n}{\Delta t}, v^h) + b^*(u^h_{n+1}, u^h_{n+1}, v^h)\right) + M^{-2}(\nabla u^h_{n+1}, \nabla v^h) - (p^h_{n+1}, \nabla.v^h) \tag{1.5}$$
$$+(-\nabla\phi^h_{n+1} + u^h_{n+1}\times B, v^h\times B) = (f_{n+1}, v^h), \quad \forall v^h \in X^h,$$
$$\left(\nabla\cdot u^h_{n+1}, q^h\right) = 0, \ \ \forall q^h \in Q^h, \tag{1.6}$$
$$(\nabla\phi^h_{n+1} - u^h_{n+1}\times B, \nabla\psi^h) = 0, \ \forall\psi^h \in S^h, \tag{1.7}$$
$$u^h(x,0) = u^h_0(x). \tag{1.8}$$

The problem restricted $v^h \in V^h$ in (1.5)-(1.8): find $(u^h_{n+1}, p^h_{n+1}, \phi^h_{n+1}) \in V^h \times Q^h \times S^h$ for each $n = 0,1,2,\ldots,K-1$ that satisfies (1.7)-(1.8) and

$$N^{-1}\left((\frac{u^h_{n+1}-u^h_n}{\Delta t}, v^h) + b^*(u^h_{n+1}, u^h_{n+1}, v^h)\right) + M^{-2}(\nabla u^h_{n+1}, \nabla v^h) \tag{1.9}$$
$$+(-\nabla\phi^h_{n+1} + u^h_{n+1}\times B, v^h\times B) = (f_{n+1}, v^h), \quad \forall v^h \in V^h.$$

Thus, solving the problem associated with (1.9), (1.7), (1.8) is equivalent to (1.5)-(1.8). The solutions, stability and convergency analysis of the Algorithm (1.1) are presented in [23], comprehensively. In this paper, we will investigate the numerical regularization for SMHD with linear time relaxation term which is introduced SMHDLTRM.

This paper is organized as follows. We give some necessary definitions and lemmas in Section 2. All definitions and lemmas are referred by [12]. The stability and convergency analysis of SMHDLTRM with BE method is given comprehensively in Section 3. In Section 4, numerical tests are given to illustrate the theoretical results as well. The results are compared with BE and CN methods for SMHD. Finally, the conclusions are given in the last section.

2. Notation and Preliminaries

We will denote the L^2-norm and inner product by $(.,.)$ and $\|.\|$, respectively. The $W^k_p(\Omega)-$norm and the $W^k_p(\Omega)-$semi-norm are denoted by $\|.\|_{p,k} := \|.\|_{W^k_p(\Omega)}$ and $|.|_{W^k_p(\Omega)}$, respectively. For $p = 2$, we will write $H^k(\Omega) := W^k_2(\Omega)$ and denote $\|.\|_k$ and $|.|_k$ for the corresponding norm and semi-norm. Denote the pressure, velocity and electric potential spaces by

$$Q := \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\},$$
$$X := \left\{ v \in H^1(\Omega)^d : v|_{\partial\Omega} = 0 \right\},$$
$$S := \left\{ \psi \in H^1(\Omega) : \psi|_{\partial\Omega} = 0 \right\}$$

respectively. $X^* = H^{-1}(\Omega)$ is the closure of $L^2(\Omega)$ in $\|.\|_{-1}$, where

$$\|f\|_{-1} := \sup_{v\in X} \frac{(f,v)}{\|\nabla v\|}.$$

Let $L^q(0,T;W_p^k(\Omega))$ denote the space

$$L^q(0,T;W_p^k(\Omega)) = \left\{ \begin{array}{c} v:(0,T)\to W_p^k(\Omega): v \text{ is measurable} \\ \text{and } \int_0^T \|v(t)\|^q_{W_p^k(\Omega)}\,dt < \infty \end{array} \right\}$$

endowed with the norm

$$\|v\|_{L^q(0,T;W_p^k(\Omega))} := \left(\int_0^T \|v(t)\|^q_{W_p^k(\Omega)}\,dt\right)^{1/q}.$$

Write $L^q(W_p^k) = L^q(0,T;W_p^k(\Omega))$ and $C^m(W_p^k) = C^m([0,T];W_p^k(\Omega))$. For $v(x,t)$ and $1\le p\le\infty$, we introduce

$$\|v\|_{\infty,k} := ess \sup_{0<t<T} \|v(t,\cdot)\|_k\,,$$

$$\|v\|_{p,k} := \left(\int_0^T \|v(t,\cdot)\|_k^p\,dt\right)^{1/p}.$$

Let V be the divergence free subspace of X, i.e.

$$V = \{v\in X : (q,\nabla\cdot v) = 0, \forall q\in Q\}\,.$$

Definition 2.1. Skew-symmetric trilinear form $b^* : X\times X\times X\to\mathbb{R}$ is defined as

$$b^*(u,v,w) = \frac{1}{2}\,(u\cdot\nabla v,w) - \frac{1}{2}\,(u\cdot\nabla w,v)\,.$$

Lemma 2.2. *Let $\Omega\subset\mathbb{R}^2$ or $\mathbb{R}^3$. For all $u,v,w\in X$*

$$b^*(u,v,w) \le C(\Omega)\,\|\nabla u\|\,\|\nabla v\|\,\|\nabla w\|$$

and

$$b^*(u,v,w) \le C(\Omega)\,\|u\|^{\frac{1}{2}}\,\|\nabla u\|^{\frac{1}{2}}\,\|\nabla v\|\,\|\nabla w\|\,.$$

Lemma 2.3 (Discrete Gronwall Lemma)**.** *Let $\Delta t, B, a_n, b_n, c_n, d_n$ for integers $n\ge 0$ be nonnegative numbers such that*

$$a_l + \Delta t\sum_{n=0}^{l} b_n \le \Delta t\sum_{n=0}^{l} d_n a_n + \Delta t\sum_{n=0}^{l} c_n + B \;\; for\; l\ge 0.$$

Suppose that $\Delta t d_n < 1$ for each n. Then,

$$a_l + \Delta t\sum_{n=0}^{l} b_n \le \left(\Delta t\sum_{n=0}^{l} c_n + B\right)\exp\left(\Delta t\sum_{n=0}^{l}\frac{d_n}{1-\Delta t d_n}\right) for\; l\ge 0.$$

Let τ^h be a uniformly regular triangulation of Ω and

$$h = \sup_{K\in\tau^h} d(K).$$

Let $X^h\subset X$, $Q^h\subset Q$ and $S^h\subset S$ be a conforming velocity, pressure, potential mixed finite element space which satisfy the LBB^h condition,

$$\inf_{q\in Q^h}\sup_{v\in X^h}\frac{(q,\nabla\cdot v)}{|v|_1\,\|q\|} \ge C > 0.$$

Let V^h be as follows:

$$V^h = \left\{v\in X^h : \int_\Omega q\nabla\cdot v = 0 \quad \forall q\in Q^h\right\}.$$

Note that in general $V^h \not\subset V$. We assume that the velocity, pressure, potential spaces satisfy the LBB^h condition and the following approximation properties

$$\inf_{v\in X^h}\|u-v\| \le Ch^{k+1}\,\|u\|_{k+1} \;\;,\;\; u\in H^{k+1}(\Omega)^d,$$

$$\inf_{v\in X^h}\|u-v\|_1 \le Ch^k\,\|u\|_{k+1}\,,\ u\in H^{k+1}(\Omega)^d,$$
$$\inf_{\psi\in X^h}\|\phi-\psi\|_1 \le Ch^r\|\phi\|_{r+1},\ \phi\in H^{r+1}(\Omega),$$
$$\inf_{r\in Q^h}\|p-r\| \le Ch^{s+1}\,\|p\|_{s+1}\,,\ p\in H^{s+1}(\Omega).$$

If LBB^h condition holds, the following inequality which will be used in the proof will be satisfied for all $u\in V$:

$$\inf_{v\in V^h}\|\nabla(u-v)\| \le C(\Omega)\inf_{v\in X^h}\|\nabla(u-v)\|\,.$$

3. SMHD Equations with Linear Time Relaxation

Let SMHDLTRM with Backward Euler discretization is considered with the following algorithm.

Theorem 3.1 (Algorithm)**.** *Given $u_0\in V$, find $(u^h_{n+1},p^h_{n+1},\phi^h_{n+1})\in X^h\times Q^h\times S^h$ for each $n=0,1,2,\ldots,K-1$, satisfying*

$$\begin{gathered}
N^{-1}\left((\frac{u^h_{n+1}-u^h_n}{\Delta t},v^h)+b^*(u^h_{n+1},u^h_{n+1},v^h)\right)+M^{-2}(\nabla u^h_{n+1},\nabla v^h)\\
-(p^h_{n+1},\nabla.v^h)+(-\nabla\phi^h_{n+1}+u^h_{n+1}\times B,v^h\times B)+\\
\kappa\left(u^h_{n+1-}\bar{u}^h_{n+1},v^h\right)=(f_{n+1},v^h),\quad \forall v^h\in X^h,\\
\left(\nabla\cdot u^h_{n+1},q^h\right)=0,\ \ \forall q^h\in Q^h,\\
(\triangledown\phi^h_{n+1}-u^h_{n+1}\times B,\triangledown\psi^h)=0,\quad \forall\psi^h\in S^h,\\
\frac{\bar{u}^h_{n+1}-\bar{u}^h_n}{\Delta t}=\frac{u^h_{n+1}-\overline{u}^h_{n+1}}{\delta},\\
\bar{u}^h_0=u^h_0.
\end{gathered}$$

The stability analysis of the Algorithm 3.1 is given with the following theorem.

3.1. Stability of the Backward-Euler Method for SMHDLTRM.

Theorem 3.2 (Stability)**.** *The solution u^h obtained by Algorithm 3.1 is unconditionally stable and satisfies the following unconditional stability bound*

$$\begin{aligned}
&\left\|u^h_M\right\|^2+N\kappa\delta\left\|\overline{u}^h_M\right\|^2+\sum_{n=0}^{M-1}\left\|u^h_{n+1}-u^h_n\right\|^2+\\
&N(\kappa\delta+\frac{2\kappa\delta^2}{\Delta t})\sum_{n=0}^{M-1}\left\|\overline{u}^h_{n+1}-\overline{u}^h_n\right\|^2+\frac{2\Delta tN}{M^2}\sum_{i=1}^{M-1}\left\|\nabla u^h_{n+1}\right\|^2\\
&+2\Delta tN\sum_{n=0}^{M-1}\left\|j^h_{n+1}\right\|^2\\
&\le\left\|u^h_0\right\|^2+N\kappa\delta\left\|\overline{u}^h_0\right\|^2+2\Delta tM^2\sum_{n=0}^{M-1}\left\|f^h_{n+1}\right\|^2_{-1}.
\end{aligned}\tag{3.1}$$

and also

$$\max_{0\le n\le K-1}\left\|\nabla\phi^h_{n+1}\right\|^2\le\overline{B}^2\left\|u^h\right\|^2_{L^\infty(L^2)}$$

or

$$\max_{0\le n\le M-1}\left\|\nabla\phi^h_{n+1}\right\|^2\le\overline{B}^2\left(\left\|u^h_0\right\|^2+M^2N\Delta t\sum_{i=1}^{M-1}\|f_{n+1}\|^2_{-1}\right)\tag{3.2}$$

where

$$\overline{B}:=\|B\|_{L^\infty(L^\infty)}\,.$$

Proof. Putting $v^h = u^h_{n+1}$,$q^h = p^h_{n+1}$,$\psi^h = \phi^h_{n+1}$ into Algorithm 3.1 gives as

$$N^{-1}\left((\frac{u^h_{n+1}-u^h_n}{\Delta t}, u^h_{n+1}) + b^*(u^h_{n+1}, u^h_{n+1}, u^h_{n+1})\right) + M^{-2}(\nabla u^h_{n+1}, \nabla u^h_{n+1})$$
$$-(p^h_{n+1}, \nabla . u^h_{n+1}) + (-\nabla\phi^h_{n+1} + u^h_{n+1}\times B, u^h_{n+1}\times B)+$$
$$\kappa\left(u^h_{n+1}-\bar{u}^h_{n+1}, u^h_{n+1}\right) = (f_{n+1}, u^h_{n+1}), \tag{3.3}$$
$$\left(\nabla\cdot u^h_{n+1}, p^h_{n+1}\right) = 0,$$
$$(-\nabla\,\phi^h_{n+1} + u^h_{n+1}\times B, -\nabla\,\phi^h_{n+1}) = 0, \tag{3.4}$$
$$\frac{\bar{u}^h_{n+1}-\bar{u}^h_n}{\Delta t} = \frac{u^h_{n+1}-\bar{u}^h_{n+1}}{\delta}, \tag{3.5}$$
$$\bar{u}^h_0 = u^h_0.$$

Since $j^h_{n+1} = -\nabla\,\phi^h_{n+1} + u^h_{n+1}\times B$ and adding (3.3) and (3.4) we obtain

$$N^{-1}(\frac{u^h_{n+1}-u^h_n}{\Delta t}, u^h_{n+1}) + M^{-2}\left\|\nabla u^h_{n+1}\right\|^2 + \left\|j^h_{n+1}\right\|^2 \tag{3.6}$$
$$\kappa\left(u^h_{n+1}-\bar{u}_{n+1}, u^h_{n+1}\right) = (f_{n+1}, u^h_{n+1})$$

from (3.5)

$$u^h_{n+1} - \bar{u}^h_{n+1} = \frac{\delta}{\Delta t}(\bar{u}^h_{n+1}-\bar{u}^h_n) \tag{3.7}$$

we substitute (3.7) into (3.6)

$$\frac{N^{-1}}{\Delta t}\left(u^h_{n+1}-u^h_n,\ u^h_{n+1}\right) + M^{-2}\left(\nabla u^h_{n+1}, \nabla u^h_{n+1}\right) \tag{3.8}$$
$$+(j^h_{n+1}, j^h_{n+1}) + \frac{\kappa\delta}{\Delta t}\left(\bar{u}^h_{n+1}-\bar{u}^h_n,\ u^h_{n+1}\right) = \left(f_{n+1}, u^h_{n+1}\right).$$

Then, adding and subtracting $\bar{u}^h_{n+1}$ from the second component of $(\bar{u}^h_{n+1}-\bar{u}^h_n, u^h_{n+1})$ we obtain

$$\begin{aligned}
\left(\bar{u}^h_{n+1}-\bar{u}^h_n,\ u^h_{n+1}\right) &= (\bar{u}^h_{n+1}-\bar{u}^h_n, u^h_{n+1}+\bar{u}^h_{n+1}-\bar{u}^h_{n+1}) \\
&= (\bar{u}^h_{n+1}-\bar{u}^h_n, \bar{u}^h_{n+1}) + (\bar{u}^h_{n+1}-\bar{u}^h_n, u^h_{n+1}-\bar{u}^h_{n+1}), \\
\left(\bar{u}^h_{n+1}-\bar{u}^h_n, \bar{u}^h_{n+1}\right) &= \frac{\left\|\bar{u}^h_{n+1}-\bar{u}^h_n\right\|^2 + \left\|\bar{u}^h_{n+1}\right\|^2 - \left\|\bar{u}^h_n\right\|^2}{2}, \\
(\bar{u}^h_{n+1}-\bar{u}^h_n, u^h_{n+1}-\bar{u}^h_{n+1}) &= (\bar{u}^h_{n+1}-\bar{u}^h_n, \frac{\delta}{\Delta t}(\bar{u}^h_{n+1}-\bar{u}^h_n)) \\
&= \frac{\delta}{\Delta t}\left\|\bar{u}^h_{n+1}-\bar{u}^h_n\right\|^2.
\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) multiplying by 2 and summing from $n = 1$ to $M-1$ gives the expected result.

$$\begin{aligned}
&\left\|u^h_M\right\|^2 + N\kappa\delta\left\|\overline{u}^h_M\right\|^2 + \sum_{n=0}^{M-1}\left\|u^h_{n+1}-u^h_n\right\|^2 + \frac{2\Delta t N}{M^2}\sum_{n=0}^{M-1}\left\|\nabla u^h_{n+1}\right\|^2 + \\
&2\Delta t N\sum_{n=0}^{M-1}\left\|j^h_{n+1}\right\|^2 + N(\kappa\delta + \frac{2\kappa\delta^2}{\Delta t})\sum_{n=0}^{M-1}\left\|\overline{u}^h_{n+1}-\overline{u}^h_n\right\|^2 \\
\leq\ &\left\|u^h_0\right\|^2 + N\kappa\delta\left\|\overline{u}^h_0\right\|^2 + 2\Delta t N M^2\sum_{n=0}^{M-1}\left\|f_{n+1}\right\|^2_{-1}.
\end{aligned}$$

Similarly, setting $\psi = \phi^h_{n+1}$ in (3.4) with Cauchy-Schwarz inequality gives

$$\left\|\nabla\phi^h_{n+1}\right\|^2 \leq \left\|u^h_{n+1}\times B\right\|^2 \leq \overline{B}^2\left\|u^h_{n+1}\right\|^2 \leq \overline{B}^2\left\|u^h\right\|^2_{L^\infty(L^2)}. \tag{3.10}$$

Apply (3.1) to (3.10) to prove (3.2). □

3.2. **Error Analysis of the Backward-Euler Method for SMHDLTRM.** The error analysis of the Algorithm 3.1 is given with the following theorem.

Theorem 3.3. *Let $(u(t), \phi(t), p(t))$ be a sufficiently smooth solution of (1.4).Suppose $(u_0^h(t), \phi_0^h(t), p_0^h(t))$ are approximations of $(u(0), \phi(0), p(0))$ to within the accuracy of the interpolant. Then there is a constant C such that*

$$\begin{aligned}
\left|\left|\left|u - u^h\right|\right|\right|_{\infty,0} &\leq Ch^{k+1}\, |||u|||_{\infty,k+1} + F(\Delta t, h),\\
\left(\Delta t \sum_{n=0}^{K-1} \left\|\nabla\phi - \nabla\phi^h\right\|^2\right)^{1/2} &\leq Ch^r\, |||\phi|||_{2,r+1} + F(\Delta t, h),\\
\left(\Delta t \sum_{n=0}^{K-1} \left\|\nabla\left(u_{n+1} - u_{n+1}^h\right)\right\|^2\right)^{1/2} &\leq Ch^k\, |||u|||_{2,k+1} + F(\Delta t, h)
\end{aligned}$$

where

$$\begin{aligned}
F(\Delta t, h) \;:=\; & N^{-1/2}\left\|A_1\bar{\gamma}_1^h - A_2\bar{\gamma}_0^h\right\| + (\frac{\kappa\delta}{2})^{1/2}\left\|\bar{\gamma}_0^h\right\|\\
&+CMh^k\, |||\bar{u}|||_{2,k+1} + CMh^{s+1}\left(|||p|||_{2,s+1}\right)\\
&+Ch^r\, |||\phi|||_{2,r+1} + C\bar{B}h^{k+1}\, |||\bar{u}|||_{2,k+1} + CMh^k\left(|||\bar{u}|||_{4,k+1}^2 + |||\nabla u|||_{4,0}^2\right)\\
&+CMh^k\left(||\,|\bar{u}|\,||_{4,k+1}^2 + \left[\left\|\bar{u}_0^h\right\| + |||f_{n+1}|||_{2,-1}\right]\right) + C\Delta t\, \|u_{tt}(t)\|_{2,0} + C\,(\kappa\delta)\, \|\bar{u}_t\|_{2,0}\,.
\end{aligned}$$

with the small time step condition $\Delta t < \frac{1}{K_n} = \left(\|B\|^2 + \|\nabla u_{n+1}\|^4 + 1\right)^{-1}$.

Proof. At time $t_n \to t_{n+1}$, true solution u satisfies

$$\begin{aligned}
N^{-1}\left((\frac{u_{n+1} - u_n}{\Delta t}, v^h) + b^*(u_{n+1}, u_{n+1}, v^h)\right) + M^{-2}(\nabla u_{n+1}, \nabla v^h)&\\
+(-\nabla\phi_{n+1} + u_{n+1} \times B, v^h \times B) - (p_{n+1}, \nabla.v^h)&\\
+\kappa\left(u_{n+1} - \bar{u}_{n+1}, v^h\right) = (f_{n+1}, v^h) - \tau(u_n, \phi_n, v^h), \quad \forall v^h \in V^h,& \qquad (3.11)\\
\left(\nabla \cdot u_{n+1}, q^h\right) = 0, \quad \forall q^h \in Q^h\,,&\\
(-\nabla\phi_{n+1} + u_{n+1} \times B, \nabla\psi^h) = 0, \quad \forall\psi^h \in S^h,& \qquad (3.12)\\
\frac{\bar{u}_{n+1} - \bar{u}_n}{\Delta t} = \frac{u_{n+1} - \bar{u}_{n+1}}{\delta}&
\end{aligned}$$

where $\tau(v)$ is the consistency error. Subtract (3.11) and (3.12) from (1.5) and (1.7) respectively and decompose the velocity, potential and current density errors:

$$\begin{aligned}
e_{u,n} &= u_n - u_n^h = \eta_n - \gamma_n^h, & \eta_n &= u_n - U_n, & \gamma_n^h &= u_n^h - U_n,\\
e_{\phi,n} &= \phi_n - \phi_n^h = \zeta_n - \Phi_n^h, & \zeta_n &= \phi_n - \widetilde{\phi}_n^h, & \Phi_n^h &= \phi_n^h - \widetilde{\phi}_n^h,\\
e_{j,n} &= j_n - j_n^h = \chi_n - J_n^h, & \chi_n &= -\nabla\,\zeta_n + \eta_n \times B, & J_n^h &= -\nabla\Phi_n^h + \gamma_n^h \times B,\\
\widetilde{e}_{j,n} &= \bar{j}_n - \bar{j}_n^h = \bar{\chi}_n - \bar{J}_n^h, & \bar{\chi}_n &= -\nabla\,\zeta_n + b_n \times B, & \bar{J}_n^h &= -\nabla\Phi_n^h + a_n^h \times B,\\
\widetilde{e}_{u,n} &= \bar{u}_n - \bar{u}_n^h = \bar{\eta}_n - \bar{\gamma}_n^h, & \bar{\eta}_n &= \bar{u}_n - \overline{U}_n, & \bar{\gamma}_n^h &= \bar{u}_n^h - \overline{U}_n,\\
a_n &= A_1\bar{\gamma}_n^h - A_2\bar{\gamma}_{n-1}^h, & b_n &= A_1\bar{\eta}_n^h - A_2\bar{\eta}_{n-1}^h.
\end{aligned}$$

Fix $\widetilde{q}_n^h \in Q^h$. Note that $(\widetilde{q}_n^h, \nabla . v^h) = 0$ for any $v^h \in V^h$. Write

$$\frac{N^{-1}}{\Delta t}\left(e_{u,n+1} - e_{u,n}, v^h\right) + R_n^h(v^h) + \left(\nabla e_{u,n+1}, \nabla v^h\right) + \left(e_{j,n+1}, v^h \times B\right) +$$

$$\frac{\kappa\delta}{\Delta t}\left(\widetilde{e}_{u,n+1} - \widetilde{e}_{u,n}, v^h\right) - (p_{n+1} - \widetilde{q}_{n+1}^h, \nabla . v^h) + \tau(u_n, \phi_n, v^h) = 0, \tag{3.13}$$

$$\left(e_{j,n+1}, - \nabla \psi^h\right) = 0 \tag{3.14}$$

where $R_n^h(v) = N^{-1}\left[b^*(u_{n+1}, u_{n+1}, v) - b^*(u_{n+1}^h, u_{n+1}^h, v)\right]$. Let rewrite the term $e_{u,n+1}$ as follows

$$e_{u,n+1} = \frac{\delta}{\Delta t}(\widetilde{e}_{u,n+1} - \widetilde{e}_{u,n}) + \widetilde{e}_{u,n+1} = \underbrace{(1 + \frac{\delta}{\Delta t})}_{A_1}\widetilde{e}_{u,n+1} - \underbrace{\frac{\delta}{\Delta t}}_{A_2}\widetilde{e}_{u,n} = A_1\widetilde{e}_{u,n+1} - A_2\widetilde{e}_{u,n} \tag{3.15}$$

and put (3.15) into (3.13) and (3.14)

$$\frac{N^{-1}}{\Delta t}\left(A_1\widetilde{e}_{u,n+1} - (A_1 + A_2)\widetilde{e}_{u,n} + A_2\widetilde{e}_{u,n-1}, v^h\right) + \left(\nabla(A_1\widetilde{e}_{u,n+1} - A_2\widetilde{e}_{u,n}), \nabla v^h\right)$$

$$+\left(-\nabla e_{\phi,n+1} + (A_1\widetilde{e}_{u,n+1} - A_2\widetilde{e}_{u,n}) \times B, v^h \times B\right) + \frac{\kappa\delta}{\Delta t}\left(\widetilde{e}_{u,n+1} - \widetilde{e}_{u,n}, v^h\right)$$

$$-(p_{n+1} - \widetilde{q}_{n+1}^h, \nabla . v^h) + R_n^h(v^h) + \tau(u_n, \phi_n, v^h) = 0,$$

$$\left(-\nabla e_{\phi,n+1} + (A_1\widetilde{e}_{u,n+1} - A_2\widetilde{e}_{u,n}) \times B, - \nabla \psi^h\right) = 0.$$

Set $v^h = A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h$, $\psi^h = \Phi_{n+1}^h$ to get the error equations

$$N^{-1}\left(\frac{A_1\left(\overline{\eta}_{n+1} - \overline{\gamma}_{n+1}^h\right) - (A_1 + A_2)\left(\overline{\eta}_n - \overline{\gamma}_n^h\right) + A_2\left(\overline{\eta}_{n-1} - \overline{\gamma}_{n-1}^h\right)}{\Delta t}, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right)$$

$$+M^{-2}\left(\nabla\left(A_1\left(\overline{\eta}_{n+1} - \overline{\gamma}_{n+1}^h\right) - A_2\left(\overline{\eta}_n - \overline{\gamma}_n^h\right)\right), \nabla(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h)\right)$$

$$-\left(p_{n+1} - q_{n+1}^h, \nabla\cdot(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h)\right)$$

$$+(-\nabla(\zeta_{n+1} - \Phi_{n+1}^h) + (\left(A_1\left(\bar{\eta}_{n+1} - \bar{\gamma}_{n+1}^h\right) - A_2\left(\bar{\eta}_n - \bar{\gamma}_n^h\right)\right) \times B, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h \times B)$$

$$+\kappa\delta\left(\frac{\left(\bar{\eta}_{n+1} - \bar{\gamma}_{n+1}^h\right) - \left(\bar{\eta}_n - \bar{\gamma}_n^h\right)}{\Delta t}, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right) + R_n^h(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h) =$$

$$-\tau(u_{n+1}, \bar{u}_{n+1}, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h)$$

$$\left(-\nabla(\zeta_{n+1} - \Phi_{n+1}^h) + (A_1\left(\bar{\eta}_{n+1} - \bar{\gamma}_{n+1}^h\right) - A_2\left(\bar{\eta}_n - \bar{\gamma}_n^h\right)) \times B, \nabla\Phi_{n+1}^h\right) = 0$$

by decomposing the terms;

$$\frac{N^{-1}}{\Delta t}\left(A_1\bar{\gamma}_{n+1}^h - (A_1 + A_2)\bar{\gamma}_n^h + A_2\bar{\gamma}_{n-1}^h, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right) +$$

$$M^{-2}\left(\nabla(A_1\bar{\gamma}_{n+1}^h - A_2\bar{\gamma}_n^h), \nabla\left(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right)\right)$$

$$+\left(-\nabla\Phi_{n+1}^h + (A_1\bar{\gamma}_{n+1}^h - A_2\bar{\gamma}_n^h) \times B, (A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h) \times B\right) +$$

$$\frac{\kappa\delta}{\Delta t}\left(\bar{\gamma}_{n+1}^h - \bar{\gamma}_n^h, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right)$$

$$= \frac{N^{-1}}{\Delta t}\left(A_1\bar{\eta}_{n+1} - (A_1 + A_2)\bar{\eta}_n + A_2\bar{\eta}_{n-1}, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right) +$$

$$M^{-2}\left(\nabla(A_1\bar{\eta}_{n+1} - A_2\bar{\eta}_n), \nabla\left(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right)\right)$$

$$+\left(-\nabla\zeta_{n+1}^h + (A_1\bar{\eta}_{n+1} - A_2\bar{\eta}_n) \times B, \left(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right) \times B\right) +$$

$$\frac{\kappa\delta}{\Delta t}\left(\bar{\eta}_{n+1} - \bar{\eta}_n, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right) - R_n^h(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h) +$$

$$\left(p_{n+1} - q_{n+1}^h, \nabla\cdot\left(A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h\right)\right) - \tau(u_{n+1}, \bar{u}_{n+1}, A_1\overline{\gamma}_{n+1}^h - A_2\overline{\gamma}_n^h) \tag{3.16}$$

and

$$((-\nabla\Phi_{n+1}^h + (A_1\bar{\gamma}_{n+1}^h - A_2\bar{\gamma}_n^h) \times B), \nabla\Phi_{n+1}^h) = ((-\nabla\zeta_{n+1} + (A_1\bar{\eta}_{n+1} - A_2\bar{\eta}_n) \times B, \nabla\Phi_{n+1}^h).$$

Set $A_1\overline{\gamma}^h_{n+1} - A_2\overline{\gamma}^h_n = a_{n+1}$ and $A_1\bar{\eta}_{n+1} - A_2\bar{\eta}_n = b_{n+1}$ in (3.16) and in (3.2)

$$\frac{N^{-1}}{\Delta t}(a_{n+1} - a_n, a_{n+1}) + M^{-2}(\nabla a_{n+1}, \nabla a_{n+1}) + \left(\bar{J}^h_{n+1}, a_{n+1} \times B\right)$$
$$+\frac{\kappa\delta}{\Delta t}\left(\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n, a_{n+1}\right) = \frac{N^{-1}}{\Delta t}(b_{n+1} - b_n, a_{n+1}) + M^{-2}(\nabla b_{n+1}, \nabla a_{n+1})$$
$$+(\overline{\chi}_{n+1}, a_{n+1} \times B) + \frac{\kappa\delta}{\Delta t}(\bar{\eta}_{n+1} - \bar{\eta}_n, a_{n+1}) - R^h_n(a_{n+1})$$
$$+\left(p_{n+1} - \tilde{q}^h_{n+1}, \nabla \cdot a_{n+1}\right) - \tau(u_{n+1}, \bar{u}_{n+1}, a_{n+1}), \tag{3.17}$$
$$(\overline{J}^h_{n+1}, \nabla\Phi^h_{n+1}) = (\overline{\chi}_{n+1}, \nabla\Phi^h_{n+1}). \tag{3.18}$$

Let $U_n, \widetilde{\phi}^h_n$ be the L^2 projections of u and ϕ so that $(b_{n+1} - b_n, a_{n+1}) = 0$. Fix $\widetilde{q}^h_n \in Q^h$. Note that $(\widetilde{q}^h_n, \nabla.v) = 0$ for any $v \in V^h$. Thus, subtracting (3.17) from (3.18) we obtain the equation as follows

$$\frac{N^{-1}}{\Delta t}(a_{n+1} - a_n, a_{n+1}) + M^{-2}(\nabla a_{n+1}, \nabla a_{n+1}) + \frac{\kappa\delta}{\Delta t}\left(\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n, a_{n+1}\right)$$
$$+\left(\overline{J}^h_{n+1}, -\nabla\Phi^h_{n+1} + a_{n+1} \times B\right)$$
$$= \frac{N^{-1}}{\Delta t}(b_{n+1} - b_n, a_{n+1}) + M^{-2}(\nabla b_{n+1}, \nabla a_{n+1}) +$$
$$\left(\overline{\chi}_{n+1}, -\nabla\phi^h_{n+1} + a_{n+1} \times B\right)$$
$$+\frac{\kappa\delta}{\Delta t}(\bar{\eta}_{n+1} - \bar{\eta}_n, a_{n+1}) - R^h_n(a_{n+1})$$
$$+\left(p_{n+1} - \tilde{q}^h_{n+1}, \nabla \cdot a_{n+1}\right) - \tau(u_{n+1}, \bar{u}_{n+1}, a_{n+1}). \tag{3.19}$$

From (3.18)

$$\left\|\nabla\Phi^h_{n+1}\right\|^2 = \left(a_{n+1} \times B - \overline{\chi}_{n+1}, \nabla\Phi^h_{n+1}\right) \tag{3.20}$$

and adding (3.20) to (3.19);

$$\frac{N^{-1}}{\Delta t}(a_{n+1} - a_n, a_{n+1}) + M^{-2}\|\nabla a_{n+1}\|^2 + \left\|\overline{J}^h_{n+1}\right\|^2$$
$$+\left\|\nabla\Phi^h_{n+1}\right\|^2 + \frac{\kappa\delta}{\Delta t}\left(\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n, a_{n+1}\right) =$$
$$\frac{N^{-1}}{\Delta t}(b_{n+1} - b_n, a_{n+1}) + M^{-2}(\nabla b_{n+1}, \nabla a_{n+1})$$
$$\left(\bar{\chi}_{n+1}, -\nabla\Phi^h_{n+1} + a_{n+1} \times B\right) +$$
$$\frac{\kappa\delta}{\Delta t}(\bar{\eta}_{n+1} - \bar{\eta}_n, a_{n+1}) - R^h_n(a_{n+1}) + \left(p_{n+1} - \tilde{q}^h_{n+1}, \nabla \cdot a_{n+1}\right)$$
$$\left(a_{n+1} \times B - \overline{\chi}_{n+1}, \nabla\Phi^h_{n+1}\right) - \tau(u_{n+1}, \bar{u}_{n+1}, a_{n+1}). \tag{3.21}$$

Now, let norm the $\frac{\kappa\delta}{\Delta t}\left(\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n, a_{n+1}\right)$ term on the left hand side of (3.21) as follows

$$\frac{\kappa\delta}{\Delta t}\left(\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n, A_1\bar{\gamma}^h_{n+1} - A_2\bar{\gamma}^h_n\right) = \frac{\kappa\delta}{\Delta t}\left(A_1\left\|\bar{\gamma}^h_{n+1}\right\|^2 - A_1(\bar{\gamma}^h_n, \bar{\gamma}^h_{n+1}) - A_2(\bar{\gamma}^h_n, \bar{\gamma}^h_{n+1}) + A_2\left\|\bar{\gamma}^h_n\right\|^2\right)$$
$$= \frac{\kappa\delta}{\Delta t}\left(A_1\left\|\bar{\gamma}^h_{n+1}\right\|^2 - (A_1 + A_2)\left(\frac{\left\|\bar{\gamma}^h_{n+1}\right\|^2}{2} + \frac{\left\|\bar{\gamma}^h_n\right\|^2}{2} - \frac{\left\|\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n\right\|^2}{2}\right) + A_2\left\|\bar{\gamma}^h_n\right\|^2\right)$$
$$= \frac{\kappa\delta}{\Delta t}\left(\left\|\bar{\gamma}^h_{n+1}\right\|^2\frac{(A_1 - A_2)}{2} + \left\|\bar{\gamma}^h_n\right\|^2\frac{(A_2 - A_1)}{2} + \left\|\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n\right\|^2\frac{(A_1 + A_2)}{2}\right)$$
$$= \frac{\kappa\delta}{\Delta t}\left(\frac{\left\|\bar{\gamma}^h_{n+1}\right\|^2 - \left\|\bar{\gamma}^h_n\right\|^2}{2} + \left(A_2 + \frac{1}{2}\right)\left\|\bar{\gamma}^h_{n+1} - \bar{\gamma}^h_n\right\|^2\right).$$

Put them in (3.21)

$$
\begin{aligned}
&\frac{N^{-1}}{\Delta t}\left(\frac{\|a_{n+1}\|^2-\|a_n\|^2+\|a_{n+1}-a_n\|^2}{2}\right)+M^{-2}\|\nabla a_{n+1}\|^2+\left\|\nabla\Phi^h_{n+1}\right\|^2\\
&+\left\|\overline{J}^h_{n+1}\right\|^2+\frac{\kappa\delta}{\Delta t}\left(\frac{\left\|\bar{\gamma}^h_{n+1}\right\|^2-\left\|\bar{\gamma}^h_n\right\|^2}{2}\right)+\frac{\kappa\delta}{\Delta t}\left(A_2+\frac{1}{2}\right)\left\|\bar{\gamma}^h_{n+1}-\bar{\gamma}^h_n\right\|^2\\
&=M^{-2}\left(\nabla b_{n+1},\nabla a_{n+1}\right)+\left(\bar{\chi}_{n+1},\bar{J}^h_{n+1}\right)+\left(p_{n+1}-\tilde{q}^h_{n+1},\nabla\cdot a_{n+1}\right)\\
&+\left(a_{n+1}\times B_{n+1}-\overline{\chi}_{n+1},\nabla\Phi^h_{n+1}\right)-R^h_n(a_{n+1})-\tau(u_{n+1},\bar{u}_{n+1},a_{n+1}).
\end{aligned}\tag{3.22}
$$

Now, let us bound the inner products on the right hand side by using Cauchy–Schwarz and Young inequalities;

$$
\begin{aligned}
\left(p_{n+1}-\tilde{q}^h_{n+1},\nabla\cdot a_{n+1}\right) &\le \left\|p_{n+1}-\tilde{q}^h_{n+1}\right\|\|\nabla\cdot a_{n+1}\|\\
&\le \frac{\varepsilon_1}{2}\left\|p_{n+1}-\tilde{q}^h_{n+1}\right\|^2+\frac{1}{2\varepsilon_1}\|\nabla a_{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
M^{-2}\left(\nabla b_{n+1},\nabla a_{n+1}\right) &\le \|\nabla b_{n+1}\|\|\nabla a_{n+1}\|\\
&\le \frac{M^{-2}\varepsilon_2}{2}\|\nabla b_{n+1}\|^2+\frac{M^{-2}}{2\varepsilon_2}\|\nabla a_{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\left(\bar{\chi}_{n+1},\bar{J}^h_{n+1}\right) &\le \|\bar{\chi}_{n+1}\|\left\|\bar{J}^h_{n+1}\right\|\\
&\le \frac{\varepsilon_3}{2}\|\bar{\chi}_{n+1}\|^2+\frac{1}{2\varepsilon_3}\left\|\bar{J}^h_{n+1}\right\|^2,
\end{aligned}
$$

$$
\left(a_{n+1}\times B_{n+1}-\overline{\chi}_{n+1},\nabla\Phi^h_{n+1}\right)\le\frac{\varepsilon_4}{2}\|a_{n+1}\|^2\|B\|^2+\frac{\varepsilon_5}{2}\left\|\overline{\chi}_{n+1}\right\|^2+\left(\frac{1}{2\varepsilon_4}+\frac{1}{2\varepsilon_5}\right)\left\|\nabla\Phi^h_{n+1}\right\|^2.
$$

for the nonlinear terms

$$
\begin{aligned}
R^h_n(\gamma^h_{n+1}) &= b^*\left(u_{n+1},u_{n+1},a_{n+1}\right)-b^*\left(u^h_{n+1},u^h_{n+1},a_{n+1}\right)\\
&= b^*\left(u_{n+1},u_{n+1},a_{n+1}\right)-b^*\left(u^h_{n+1},u_{n+1},a_{n+1}\right)\\
&+ b^*\left(u^h_{n+1},u_{n+1},a_{n+1}\right)-b^*\left(u^h_{n+1},u^h_{n+1},a_{n+1}\right)\\
&= b^*\left(e_{n+1},u_{n+1},a_{n+1}\right)+b^*\left(u^h_{n+1},e_{n+1},a_{n+1}\right)\\
&= b^*\left(A_1\widetilde{e}_{n+1}-A_2\tilde{e}_n,u_{n+1},a_{n+1}\right)+b^*\left(u^h_{n+1},A_1\widetilde{e}_{n+1}-A_2\tilde{e}_n,a_{n+1}\right)\\
&= b^*\left(b_{n+1}-a_{n+1},u_{n+1},a_{n+1}\right)+b^*\left(u^h_{n+1},b_{n+1}-a_{n+1},a_{n+1}\right)\\
&= b^*\left(b_{n+1},u_{n+1},a_{n+1}\right)-b^*\left(a_{n+1},u_{n+1},a_{n+1}\right)\\
&+ b^*\left(u^h_{n+1},b_{n+1},a_{n+1}\right)-b^*\left(u^h_{n+1},a_{n+1},a_{n+1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
b^*\left(b_{n+1},u_{n+1},a_{n+1}\right) &\le \|b_{n+1}\|^{1/2}\|\nabla b_{n+1}\|^{1/2}\|\nabla u_{n+1}\|\|\nabla a_{n+1}\|\\
&\le \frac{\varepsilon_6}{2}\|b_{n+1}\|\|\nabla b_{n+1}\|\|\nabla u_{n+1}\|^2+\frac{1}{2\varepsilon_6}\|\nabla a_{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
b^*\left(a_{n+1},u_{n+1},a_{n+1}\right) &\le \|a_{n+1}\|^{1/2}\|\nabla a_{n+1}\|^{1/2}\|\nabla u_{n+1}\|\|\nabla a_{n+1}\|\\
&\le \frac{\varepsilon_7}{2}\|a_{n+1}\|^2\|\nabla u_{n+1}\|^4+\frac{1}{2\varepsilon_7}\|\nabla a_{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
b^*\left(u^h_{n+1},b_{n+1},a_{n+1}\right) &\le \left\|u^h_{n+1}\right\|^{1/2}\left\|\nabla u^h_{n+1}\right\|^{1/2}\|\nabla b_{n+1}\|\|\nabla a_{n+1}\|\\
&\le \frac{\varepsilon_8}{2}\left\|u^h_{n+1}\right\|\left\|\nabla u^h_{n+1}\right\|\|\nabla b_{n+1}\|^2+\frac{1}{2\varepsilon_8}\|\nabla a_{n+1}\|^2.
\end{aligned}
$$

Combining these bounds gives

$$
\begin{aligned}
R_n^h(\gamma_{n+1}^h) \quad \leq \quad & \frac{\varepsilon_6}{2}\|b_{n+1}\|\|\nabla b_{n+1}\|\|\nabla u_{n+1}\|^2 \\
& +\frac{\varepsilon_7}{2}\|a_{n+1}\|^2\|\nabla u_{n+1}\|^4 \\
& +\frac{\varepsilon_8}{2}\left\|u_{n+1}^h\right\|\left\|\nabla u_{n+1}^h\right\|\|\nabla b_{n+1}\|^2+\frac{1}{\varepsilon_9}\|\nabla a_{n+1}\|^2
\end{aligned}
$$

where $\varepsilon_9 = \frac{1}{2\varepsilon_6} + \frac{1}{2\varepsilon_7} + \frac{1}{2\varepsilon_8}$.Let Δt be a fixed positive real number. If $u = u(t)$ is smooth enough, then $\tau(u_{n+1}, \bar{u}_{n+1}, \gamma_{n+1}^h)$ is the truncation error stated as follows;

$$
\begin{aligned}
\tau(u_{n+1}, \bar{u}_{n+1}, a_{n+1}) \quad &= \quad N^{-1}\left(\frac{u_{n+1}-u_n}{\Delta t}-u_t(t_{n+1})+\kappa\delta\frac{\bar{u}_{n+1}-\bar{u}_n}{\Delta t}, a_{n+1}\right) \\
&= \quad N^{-1}\left[\left(\frac{u_{n+1}-u_n}{\Delta t}-u_t(t_{n+1}), a_{n+1}\right)+\left(\kappa\delta\frac{\bar{u}_{n+1}-\bar{u}_n}{\Delta t}, a_{n+1}\right)\right] \\
&\leq \quad N^{-1}\left\|\frac{u_{n+1}-u_n}{\Delta t}-u_t(t_{n+1})\right\|\|a_{n+1}\|+N^{-1}\left\|\kappa\delta\frac{\bar{u}_{n+1}-\bar{u}_n}{\Delta t}\right\|\|a_{n+1}\| \\
&\leq \quad N^{-1}\frac{\varepsilon_{10}}{2}\left\|\frac{u_{n+1}-u_n}{\Delta t}-u_t(t_{n+1})\right\|^2+\frac{1}{2\varepsilon_{10}}\|a_{n+1}\|^2 \\
& \qquad +N^{-1}\kappa^2\delta^2\frac{\varepsilon_{11}}{2}\left\|\frac{\bar{u}_{n+1}-\bar{u}_n}{\Delta t}\right\|^2+\frac{1}{2\varepsilon_{11}}\|a_{n+1}\|^2 \\
&\leq \quad C\Delta t\int_{t^n}^{t^{n+1}}\|u_{tt}(t)\|^2\,dt+\kappa^2\delta^2\frac{C}{\Delta t}\int_{t_n}^{t_{n+1}}\|\bar{u}_t(t)\|^2\,dt+C\Delta t\frac{1}{\varepsilon_{12}}\|a_{n+1}\|^2
\end{aligned}
$$

where $\varepsilon_{11} = \frac{1}{2\varepsilon_{10}} + \frac{1}{2\varepsilon_{11}}$.Take $\varepsilon_3 = \varepsilon_5 = 1$, $\varepsilon_4 = \varepsilon_2 = 2$, $\varepsilon_1 = \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 8M^2$ all the terms on the right hand side are put in (3.22) and multiplied by 2 and Δt , then we get

$$
\begin{gathered}
N^{-1}\left(\|a_{n+1}\|^2-\|a_n\|^2+\|a_{n+1}-a_n\|^2\right)+\Delta t M^{-2}\|\nabla a_{n+1}\|^2+\Delta t\left\|\nabla\Phi_{n+1}^h\right\|^2 \\
+\Delta t\left\|\bar{J}_{n+1}^h\right\|^2+\kappa\delta\left(\left\|\bar{\gamma}_{n+1}^h\right\|^2-\left\|\bar{\gamma}_n^h\right\|^2\right)+\kappa\delta\left(2A_2+1\right)\left\|\bar{\gamma}_{n+1}^h-\bar{\gamma}_n^h\right\|^2 \\
\leq C\Delta t\left\|p_{n+1}-\tilde{q}_{n+1}^h\right\|^2+C\Delta t\|\nabla b_{n+1}\|^2+C\Delta t\left\|\overline{\chi}_{n+1}\right\|^2 \\
+C\Delta t\|b_{n+1}\|\|\nabla b_{n+1}\|\|\nabla u_{n+1}\|^2+C\Delta t\left\|u_{n+1}^h\right\|\left\|\nabla u_{n+1}^h\right\|\|\nabla b_{n+1}\|^2+\|a_{n+1}\|^2\|B\|^2 \\
+C\Delta t\|a_{n+1}\|^2\|\nabla u_{n+1}\|^4+C\Delta t^2\int_{t^n}^{t^{n+1}}\|u_{tt}(t)\|^2\,dt+C\kappa^2\delta^2\int_{t_n}^{t_{n+1}}\|\bar{u}_t(t)\|^2\,dt+C\Delta t\|a_{n+1}\|^2 .
\end{gathered} \tag{3.23}
$$

In (3.23) summing from $n = 1$ to $K-1$ gives

$$
\begin{aligned}
N^{-1}\|a_K\|^2 + \kappa\delta \left\|\bar{\gamma}_K^h\right\|^2 + \Delta t M^{-2}\sum_{n=0}^{K-1}\|\nabla a_{n+1}\|^2 + \Delta t\sum_{n=0}^{K-1}\left\|\bar{J}_{n+1}^h\right\|^2 + \Delta t\sum_{n=0}^{K-1}\left\|\nabla\Phi_{n+1}^h\right\|^2 \\
+N^{-1}\sum_{n=0}^{K-1}\|a_{n+1}-a_n\|^2 + \kappa\delta\,(2A_2+1)\sum_{n=0}^{K-1}\left\|\bar{\gamma}_{n+1}^h - \bar{\gamma}_n^h\right\|^2 \\
\leq N^{-1}\|a_0\|^2 + \kappa\delta\left\|\bar{\gamma}_0^h\right\|^2 + C\Delta t\sum_{n=0}^{K-1}\|\nabla b_{n+1}\|^2 \\
+C\Delta t\sum_{n=0}^{K-1}\left(\|B\|^2 + \|\nabla u_{n+1}\|^4 + 1\right)\|a_{n+1}\|^2 + C\Delta t\sum_{n=0}^{K-1}\left\|p_{n+1} - \tilde{q}_{n+1}^h\right\|^2 \\
+C\Delta t\sum_{n=0}^{K-1}\left\|\overline{\chi}_{n+1}\right\|^2 + C\Delta t\sum_{n=0}^{K-1}\|b_{n+1}\|\,\|\nabla b_{n+1}\|\,\|\nabla u_{n+1}\|^2 \\
+C\Delta t\sum_{n=0}^{K-1}\left\|u_{n+1}^h\right\|\left\|\nabla u_{n+1}^h\right\|\|\nabla b_{n+1}\|^2 \\
+C\Delta t^2\sum_{n=0}^{K-1}\int_{t^n}^{t^{n+1}}\|u_{tt}(t)\|^2\,dt + C\kappa^2\delta^2\sum_{n=0}^{K-1}\int_{t_n}^{t_{n+1}}\|\bar{u}_t(t)\|^2\,dt.
\end{aligned}
\tag{3.24}
$$

Therefore,

$$
\begin{aligned}
C\Delta t\sum_{n=0}^{K-1}\|\nabla b_{n+1}\|^2 &\leq C\Delta t\sum_{n=0}^{K}\|\nabla b_n\|^2 \leq C\Delta t\sum_{n=0}^{K}h^{2k}|\bar{u}_n|_{k+1}^2 \\
&\leq Ch^{2k}\,|||\bar{u}|||_{2,k+1}^2\,,
\end{aligned}
$$

$$
\begin{aligned}
C\Delta t\sum_{n=0}^{K-1}\left\|p_{n+1}-\tilde{q}_{n+1}^h\right\|^2 &\leq Ch^{2s+2}\Delta t\sum_{n=0}^{K-1}\|p(t_{n+1})\|_{s+1}^2 \\
&\leq Ch^{2s+2}\left(|||p|||_{2,s+1}^2\right),
\end{aligned}
$$

$$
\begin{aligned}
C\Delta t\sum_{n=0}^{K-1}\left\|\overline{\chi}_{n+1}\right\|^2 &\leq C\Delta t\sum_{n=0}^{K-1}\|-\nabla\zeta_{n+1} + b_{n+1}\times B\|^2 \\
&\leq C\Delta t\sum_{n=0}^{K-1}\left(\|-\nabla\zeta_{n+1}\|^2 + \|b_{n+1}\times B\|^2\right) \\
&\leq Ch^{2r}\Delta t\sum_{n=0}^{K-1}\|\phi_{n+1}\|_{r+1}^2 + C\bar{B}^2h^{2k+2}\Delta t\sum_{n=0}^{K-1}\|b_{n+1}\|_{k+1}^2 \\
&\leq Ch^{2r}\,|||\phi|||_{2,r+1}^2 + C\bar{B}^2h^{2k+2} + |||\bar{u}|||_{2,k+1}^2\,,
\end{aligned}
$$

$$
\begin{aligned}
C\Delta t\sum_{n=0}^{K-1}\|b_{n+1}\|\,\|\nabla b_{n+1}\|\,\|\nabla u_{n+1}\|^2 &\leq C\Delta t h^{2k+1}\sum_{n=0}^{K}|\bar{u}_n|_{k+1}^2\,\|\nabla u_n\|^2 \\
&\leq Ch^{2k+1}\left(|||\bar{u}|||_{4,k+1}^4 + |||\nabla u|||_{4,0}^4\right),
\end{aligned}
$$

$$
\begin{aligned}
C\Delta t \sum_{n=0}^{K-1} \left\| u_{n+1}^h \right\| \left\| \nabla u_{n+1}^h \right\| \left\| \nabla b_{n+1} \right\|^2 &\leq C\Delta t \sum_{n=0}^{K-1} \left\| \nabla u_{n+1}^h \right\|^2 \left\| \nabla b_{n+1} \right\|^2 \\
&\leq C\Delta t h^{2k} \sum_{n=0}^{K-1} |\nabla u_{n+1}|_{k+1}^2 \left\| \nabla b_{n+1}^h \right\|^2 \\
&\leq C\Delta t h^{2k} \left(\sum_{n=0}^{K} |\bar{u}_n|_{k+1}^4 + \sum_{n=0}^{K} \left\| \nabla u_{n+1}^h \right\|^4 \right) \\
&\leq C h^{2k} \left(\| |\bar{u}| \|_{4,k+1}^4 + \left[\left\| \bar{u}_0^h \right\|^2 + |||f_{n+1}|||_{2,-1}^2 \right] \right),
\end{aligned}
$$

$$
C\Delta t^2 \sum_{n=0}^{K-1} \int_{t^n}^{t^{n+1}} \|u_{tt}(t)\|^2 \, dt \leq C\Delta t^2 \, \|u_{tt}(t)\|_{2,0}^2
$$

$$
C\kappa^2\delta^2 \sum_{n=0}^{K-1} \int_{t_n}^{t_{n+1}} \|\bar{u}_t(t)\|^2 \, dt \leq C \, (\kappa\delta)^2 \, \|\bar{u}_t\|_{2,0}^2 \, .
$$

Let's put all the bounded terms in place at (3.24)

$$
\begin{aligned}
& N^{-1} \|a_K\|^2 + \frac{\kappa\delta}{2} \left\| \bar{\gamma}_K^h \right\|^2 + \Delta t M^{-2} \sum_{n=0}^{K-1} \|\nabla a_{n+1}\|^2 + \Delta t \sum_{n=0}^{K-1} \left\| \bar{J}_{n+1}^h \right\|^2 \\
& + \Delta t \sum_{n=0}^{K-1} \left\| \nabla \Phi_{n+1}^h \right\|^2 + N^{-1} \sum_{n=0}^{K-1} \|a_{n+1} - a_n\|^2 + \kappa\delta \, (2A_2 + 1) \left\| \bar{\gamma}_{n+1}^h - \bar{\gamma}_n^h \right\|^2 \\
& \leq N^{-1} \|a_0\|^2 + \frac{\kappa\delta}{2} \left\| \bar{\gamma}_0^h \right\|^2 + C\Delta t \sum_{n=0}^{K-1} \left(\|B\|^2 + \|\nabla u_{n+1}\|^2 + 1 \right) \left\| a_{n+1}^h \right\|^2 \\
& + CM^2 h^{2k} \, |||\bar{u}|||_{2,k+1}^2 + CM^2 h^{2s+2} \left(|||p|||_{2,s+1}^2 \right) + C h^{2r} \, |||\phi_{n+1}|||_{2,r+1}^2 + C\bar{B}^2 h^{2k+2} + |||\bar{u}|||_{2,k+1}^2 \\
& + CM^2 h^{2k+1} \left(|||u|||_{4,k+1}^4 + |||\nabla u|||_{4,0}^4 \right) + CM^2 h^{2k} \left(\| |\bar{u}| \|_{4,k+1}^4 + \left[\left\| \bar{u}_0^h \right\|^2 + |||f_{n+1}|||_{2,-1}^2 \right] \right) \\
& + C\Delta t^2 \, \|u_{tt}(t)\|_{2,0}^2 + C \, (\kappa\delta)^2 \, \|\bar{u}_t\|_{2,0}^2
\end{aligned}
$$

where

$$
K_n := \|B\|^2 + \|\nabla u_{n+1}\|^4 + 1.
$$

Suppose that $\Delta t < \frac{1}{K_n} = \left(\|B\|^2 + \|\nabla u_{n+1}\|^4 + 1 \right)^{-1}$. Then, applying Gronwall Lemma gives

$$
\begin{aligned}
& N^{-1} \|a_K\|^2 + \kappa\delta \left\| \bar{\gamma}_K^h \right\|^2 + \Delta t M^{-2} \sum_{n=0}^{K-1} \|\nabla a_{n+1}\|^2 + \Delta t \sum_{n=0}^{K-1} \left\| \bar{J}_{n+1}^h \right\|^2 \\
& + \Delta t \sum_{n=0}^{K-1} \left\| \nabla \Phi_{n+1}^h \right\|^2 + N^{-1} \sum_{n=0}^{K-1} \|a_{n+1} - a_n\|^2 + \kappa\delta \, (2A_2 + 1) \left\| \bar{\gamma}_{n+1}^h - \bar{\gamma}_n^h \right\|^2 \\
& \leq N^{-1} \|a_0\|^2 + \frac{\kappa\delta}{2} \left\| \bar{\gamma}_0^h \right\|^2 + CM^2 h^{2k} \, |||\bar{u}|||_{2,k+1}^2 + CM^2 h^{2s+2} \left(|||p|||_{2,s+1}^2 \right) \\
& + C h^{2r} \, |||\phi|||_{2,r+1}^2 + C\bar{B}^2 h^{2k+2} \, |||\bar{u}|||_{2,k+1}^2 + CM^2 h^{2k+1} \left(|||\bar{u}|||_{4,k+1}^4 + |||\nabla u|||_{4,0}^4 \right) \\
& + CM^2 h^{2k} \left(\| |\bar{u}| \|_{4,k+1}^4 + \left[\left\| \bar{u}_0^h \right\|^2 + |||f_{n+1}|||_{2,-1}^2 \right] \right) + C\Delta t^2 \, \|u_{tt}(t)\|_{2,0}^2 + C \, (\kappa\delta)^2 \, \|\bar{u}_t\|_{2,0}^2 \, . \qquad \square
\end{aligned}
$$

4. Numerical Examples

In this section, we test Algorithm 2 which is obtained from adding linear time relaxation term to SMHD by some numerical examples to compare with BE Algorithm 1.1 and CN methods. In the first example we consider a problem that has an exact solution. In the second example, the problem has the same exact solution but the initial conditions are different from Example 1. For the last example, we consider the problem that has no exact solution. The examined problem is about a liquid metal flow which involves large Hartmann number and interaction parameter. This is the one of the important applications of MHD in laboratory and industry. All calculations are conducted by using FreeFem++. In all examples, T :final time, Δt : time step, M :Hartmann Number, Re : Reynolds Number and h: mesh.

Example 4.1. Let $\Omega = [0,\pi]^2,\ t_0 = 0,\ T = 1$ and $B = (0,0,1)$. Then, the true solution (u, p, ϕ) of (1.3) is [23]

$$\begin{aligned} u(x,y,t) &= \left(-\frac{\partial\psi(x,y)}{\partial y}, \frac{\partial\psi(x,y)}{\partial x}\right)e^{-5t}, \\ \phi(x,y,t) &= \left(\psi(x,y) + x^2 - y^2\right)e^{-5t}, \\ p(x,y,t) &= 0 \end{aligned}$$

where $\psi(x,y) = \cos(2x)\cos(2y)$ and f and $u|_{\partial\Omega}$ are obtained from the true solution.

The convergency of the method is analyzed for different values of κ and Δt , selecting $h = \frac{1}{50}$, $Re = 25$, $M = 20$, $T = 1$, $\delta = \frac{1}{10}$ in here. The results are given in Table 1 - 4 . In Table 1, BE method for SMHD is presented with $\kappa = 0$. As seen from the Table 1, while values of κ increases, the errors decrease for different values of Δt. The errors of SMHDLTRM are more smaller for all values of κ than case of $\kappa = 0$ (SMHD). In some case, the errors of SMHDLTRM are near the errors of CN method. In Table 2, the convergence rates are given according to the values obtained from Table 1. The convergency of SMHDLTRM more rapidly than SMHD with BE method for value of $\kappa = 100$ in this table. In Table 3 and 4, values of p and ϕ are compared with SMHD and SMHDLTRM for different values of κ and Δt, too. Similar results have been observed with velocity errors for pressure and potential errors.

TABLE 1. Velocity errors of SMHD and SMHDLTRM

			$\left\|u - u_n^h\right\|$		
Δt	CN	$BE(\kappa = 0)$	$\kappa = 1$	$\kappa = 10$	$\kappa = 100$
$\frac{1}{10}$	0.0184967	0.50428	0.497796	0.441858	0.0785827
$\frac{1}{20}$	0.00463094	0.26432	0.261351	0.23536	0.0423406
$\frac{1}{40}$	0.00114474	0.13523	0.133857	0.121688	0.0199278
$\frac{1}{80}$	0.00027341	0.06838	0.0677267	0.0618988	0.00910399
$\frac{1}{160}$	$0.27587e-5$	0.03437	0.0340567	0.0312139	0.00422726

TABLE 2. Convergence rates of SMHD and SMHDLTRM

			Convergence Rates		
Δt	CN	$BE(\kappa = 0)$	$\kappa = 1$	$\kappa = 10$	$\kappa = 100$
$\frac{1}{10}$					
$\frac{1}{20}$	1.998	0.932	0.930	0.909	0.892
$\frac{1}{40}$	2.016	0.967	0.965	0.952	1.087
$\frac{1}{80}$	2.066	0.984	0.983	0.976	1.130
$\frac{1}{160}$	2.123	0.992	0.992	0.988	1.107

TABLE 3. Potential errors of SMHD and SMHDLTRM

			$\left\|\phi - \phi_n^h\right\|$		
Δt	$BE(\kappa = 0)$	$\kappa = 1$	$\kappa = 10$	$\kappa = 100$	$\kappa = 1000$
$\frac{1}{10}$	0.277553	0.146518	0.243356	0.044278	0.00143201
$\frac{1}{20}$	0.145644	0.144009	0.129734	0.0237736	0.0339454
$\frac{1}{40}$	0.0745424	0.0737849	0.0670902	0.0111404	0.0777078
$\frac{1}{80}$	0.0376974	0.0373372	0.0341279	0.00506634	0.100721
$\frac{1}{160}$	0.0189512	0.0187762	0.0172098	0.00234376	0.0853134

Table 4. Pressure errors of SMHD and SMHDLTRM

			$\left\|p - p_n^h\right\|$		
Δt	$BE(\kappa = 0)$	$\kappa = 1$	$\kappa = 10$	$\kappa = 100$	$\kappa = 1000$
$\frac{1}{10}$	0.00516417	0.273995	0.00403135	0.0002044098	0.00544574
$\frac{1}{20}$	0.00157142	0.0015402	0.00128098	$8.02922e-5$	0.0023886
$\frac{1}{40}$	0.000503184	0.000495084	0.00042631	$3.24741e-5$	0.000124972
$\frac{1}{80}$	0.000182351	0.000179983	0.000159452	$1.46648e-5$	0.000829526
$\frac{1}{160}$	$7.56518e-5$	$7.4825e-5$	$6.7532e-5$	$7.08588e-6$	0.000493156

Table 5. $\kappa = 10$ and $h = 0.1$,Velocity errors of SMHDLTRM

		$\left\|u - u_n^h\right\|$		
Δt	$\delta = 1/16$	$\delta = 1/8$	$\delta = 1/4$	$\delta = 1/2$
$\frac{1}{5}$	0.819457	0.740430	0.574260	0.202664
$\frac{1}{10}$	0.453049	0.413236	0.325795	0.110177
$\frac{1}{20}$	0.235771	0.217006	0.173898	0.056162
$\frac{1}{40}$	0.117376	0.108525	0.087487	0.0251201
$\frac{1}{80}$	0.0560031	0.0518096	0.041676	0.0126573

Table 6. $\kappa = \frac{1}{\Delta t}$ and $\delta = \Delta t^2$, Velocity errors and Convergence rates of SMHDLTRM

Δt	$\|\|u - u_h\|\|$	*Convergence Rates*
$\frac{1}{5}$	0.888983	
$\frac{1}{10}$	0.498304	0.835
$\frac{1}{20}$	0.262973	0.922
$\frac{1}{40}$	0.134923	0.963
$\frac{1}{80}$	0.068503	0.978

Table 7. $\kappa = \Delta t$ and $\delta = \Delta t$, Velocity errors and Convergence rates of SMHDLTRM

Δt	$\|\|u - u_h\|\|$	*Convergence Rates*
$\frac{1}{5}$	0.908388	
$\frac{1}{10}$	0.503635	0.851
$\frac{1}{20}$	0.264256	0.931
$\frac{1}{40}$	0.135226	0.967
$\frac{1}{80}$	0.0683798	0.984

Example 4.2. Let $\Omega = [0, \pi]^2$, $t_0 = 0$, $T = 1$, $B = (0, 0, 1)$. In this example, SMHD equation is solved both CN and BE methods and SMHDLTRM is solved with BE method for $h = \frac{1}{10}$ and $\Delta t = \frac{1}{100}$, Re $= 6766$, $M = 20$, $\delta = \frac{1}{10}$ and different values of κ and T. The boundary condition on $\partial\Omega$ is inhomogeneous Dirichlet: $\mathbf{u}_h = \mathbf{u}$. The initial data is given by

$$\begin{aligned} u_0(x, y) &= (y, -x, 0), \\ \phi_0(x, y) &= xy. \end{aligned}$$

and $\mathbf{f}$ is the same as in Example 4.1.

In this example, CN method and BE method for SMHD have big velocity errors [23]. We consider SMHDLTRM for this example to determine how the linear time relaxation effects to the solutions. As seen from the Table 9, the velocity errors of SMHDLTRM are more smaller than SMHD for all values of κ. Also, when values of κ increase, the errors decrease for different values of Δt. From here, in case the classical methods (CN or BE methods) are failed, linear time relaxation models can be applied to the problem to get more accurate solutions.

Table 8. Velocity errors of SMHD and SMHDLTRM

			$\|\|u-u_h\|\|$			
T	CN	$BE(\kappa=0)$	$\kappa=1$	$\kappa=10$	$\kappa=100$	$\kappa=1000$
1	3.49282	3.17034	3.16741	3.14119	2.89017	1.25296
2	*blowup*	2.52362	2.51909	2.47887	2.12973	0.395158
5	*blowup*	1.43754	1.43525	1.40639	0.955979	0.0126416

Table 9. Velocity errors of SMHD and SMHDLTRM

				$\|\|u-u_h\|\|$				
Δt	CN	$BE(\kappa=0)$	$\kappa=1$	$\kappa=10$	$\kappa=100$	$\kappa=1000$	$\kappa=10000$	$\kappa=100000$
1/10	3.49282	2.79728	2.77687	2.59894	1.28941	0.18971	0.129214	0.120177
1/100	4.5372	3.17034	3.16741	3.14119	2.89017	1.25296	0.203743	0.125305
1/1000	4.84189	3.36734	3.367	3.36397	3.33379	3.0466	1.26651	0.200557

5. Conclusion

In this paper, the differential filter term $\kappa(u - \overline{u})$ is added to SMHD equations for numerical regularization and thus SMHDLTRM is introduced. Moreover the stability and convergency analysis of the method are investigated. A stability and error analysis of the method are presented in Theorem 3.2 and Theorem 3.3, respectively. Also numerical experiments are conducted to verify the theoretical results. As seen from the numerical examples, the presented method obtains lower errors compared to the BE and CN for SMHD, also the errors of the method are close to the errors of CN method in some cases of κ. In the Example 4.2 the CN and the BE methods have both high velocity errors for SMHD because of the initial conditions, but the presented method provides a numerical regularization for the problem.

Consequently, the differential filter may relax the time step of the problem, thus SMHDLTRM can be used when the classical methods (BE or CN) fail to get more accurate solutions for SMHD.

Acknowledgement

This paper has been granted by the Mugla Sitki Kocman University Research Projects Coordination Office. Project Grant Number: 17/225 and title "Basitleştirilmiş Manyetohidrodinamik (MHD) denklemlerinin sonlu elemanlar metoduyla çözümleri için geliştirilen zaman rahatlatma modelleri".

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Adams, N.A., Stolz, S., Deconvolution Methods for Subgrid-Scale Approximation in LES, Modern Simulation Strategies for Turbulent Flow, R.T. Edwards, 2001.

[2] Akbas, M., Mohebujjaman, M., Rebholz, L.G., Xiao, M., *High order algebraic splitting for magnetohydrodynamics simulation*, Journal of Computational and Applied Mathematics, **321**(2017).

[3] Belenli, M.A., Kaya, S., Rebholz, L.G., Wilson, N.E., *A subgrid stabilization finite element method for incompressible magnetohydrodynamics*, Int. J. Comp. Math., **90**(7)(2013), 1506–1523.

[4] Breckling, S., Neda, M., Hill, T., *A review of time relaxation methods*, Fluids. 2 (2017), 40.

[5] Cibik, A., Eroglu, F.G., Kaya, S., *Analysis of second order time filtered backward Euler method for MHD equations*, J Sci Comput, **82**(38)(2020).

[6] Davidson, P.A., An Introduction to Magnetohydrodynamics, Cambridge University Press, United Kingdom, 2001.
[7] Erkmen, D., Kaya, S., Cibik, A., *A second order decoupled penalty projection method based on deferred correction for MHD in Elsässer variable*, Journal of Computational and Applied Mathematics, **371**(2020).
[8] Gunzburger, M.D., Meir, A.J., Peterson, J.S., *On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics*, Math. Comput., **56**(194)(1991), 523–563.
[9] Isik, O.R., *Spin up problem and accelerating convergence to steady state*, Applied Mathematical Modelling, **37**(2013), 3242–3253.
[10] Isik, O.R., Yuksel, G., Demir, B., *Analysis of second order and unconditionally stable BDF2-AB2 method for the Navier-Stokes equations with nonlinear time relaxation*, Num. Meth. Part. Diff. Equa., **34**(6)(2018), 2060–2078.
[11] Layton, W., Neda, M., *Truncation of scales by time relaxation*, Journal of Mathematical Analysis and Applications, **325**(2007), 788–807.
[12] Layton, W., Introduction to the Numerical Analysis of Incompressible Viscous Flows, SIAM, Philadelphia, 2008.
[13] Layton, W., David Pruett, C., Rebholz, Leo G., *Temporally regularized direct numerical simulation*, Appl. Math. Comput., **216**(2010), 3728–3738.
[14] Layton, W., Rebholz, L., Approximate Deconvolution Models of Turbulence, Analysis, Phenomenology and Numerical Analysis, Springer, 2012.
[15] Layton, W., Tran, H., Trenchea, C., *Numerical analysis of two partitioned methods for uncoupling evolutionary MHD flows*, Numer. Meth. Part. D. E., **30**(4)(2014), 1083–1102.
[16] Özbunar, M., Finite Element Solution of Navier-Stokes Time Relaxation Model Euler Time Discretization, M.Sc. Thesis, Grad. Sch. Nat. Appl. Sci., 2017.
[17] Pakzad, A., *On the long time behavior of time relaxation model of fluids*, arXiv:1903.12339.
[18] Peterson, J.S., *On the finite element approximation of incompressible flows of an electrically conducting fluid*, Numer. Meth. Part. D. E., **4**(1)(1988), 57–68.
[19] Rosenau, P., *Extending hydrodynamics via the regularization of the Chapman–Enskog expansion*, Phys. Rev. A., **40**(1989), 7193–7196.
[20] Schochet, S., Tadmor, E., *The regularized Chapman–Enskog expansion for scalar conservation laws*, Arch. Ration. Mech. Anal., **119**(1992), 95–107.
[21] Stolz, S., Adams, N.A., Kleiser, L., *The approximate deconvolution model for LES of compressible flows and its application to shock-turbulent-boundary layer interaction*, Phys. Fluids, **13**(2001), 2985.
[22] Yuksel, G., Ingram, R., *Numerical analysis of a finite element, Crank-Nicolson discretization for MHD flow at small magnetic Reynolds number*, Int. J. Num. Anal. Model., **10**(1)(2013), 74–98.
[23] Yuksel, G., Isik, O.R., *Numerical analysis of Backward-Euler discretization for simplified magnetohydrodynamic flows*, Appl. Math. Modell., **39**(7)(2015), 1889–1898.